College Algebra

Study Guide

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Chapter 1: Linear Functions

OVERVIEW

The sections of this chapter are:

1.1 Graphs
1.2 Visual Linear Functions
1.3 Solving for an Unknown
1.4 Algebra of Linear Functions
1.5 Systems of Equations
1.6 Interpret and Apply

In the book Making Learning Whole, the author David Perkins wrote "Math is usually taught with an overemphasis on dry, technical details, without giving students a concept of the whole game." David Perkins is a senior professor of education at the Harvard Graduate School of Education.

OBJECTIVES

By the end of the chapter, a student will be able to:

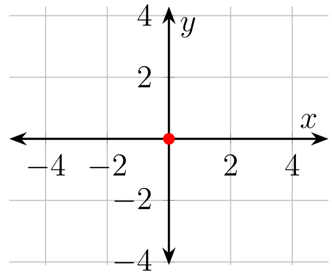
- Understand the difference between linear and nonlinear equations
- Understand the real meaning of a function
- Work with linear expressions and equations
- Interpret and apply linear functions

1.1 Graphs

The coordinate plane is also called the Cartesian Coordinate plane. The coordinate plane makes it possible to describe space in 2 dimensions. Space in 2 dimensions is infinite, or endless.

In order to measure and describe, points of reference are necessary. A coordinate plane provides points of reference and the main point of reference is the origin, that is shown, just below, with a red dot. To review, the coordinate plane consists of a horizontal axis that is labeled x, and it is called the x-axis. Notice that the x-axis has an arrow at each end. This is because space continues in that direction. Still practical use of the coordinate plane will display a sample or small piece of space. There is also a vertical axis labeled y, and it is called the y-axis. Notice that the y-axis has an arrow on each end. This is because space in 2 dimensions continues without end. Below, just to the right, one can see a **coordinate plane**. It basically consists of a horizontal x-axis, and a vertical y-axis. Below, a few characteristics of this particular graph are listed.

- The horizontal line is labeled x, and this is called the **x-axis**.
- The vertical line is labeled y, and this is called the **y-axis**.
- The **x-axis** spans from about negative 4 to about positive 4.
- The **y-axis** spans from about negative 4 to positive 4.



A student should keep in mind that the coordinate plane shown above is displaying a sample or subset of space in 2 dimensions. This is demonstrated again below. Each of the examples below, makes use of the coordinate plane, in a different way.

Fig. 1.1: $-6.5 < x < 2.5$

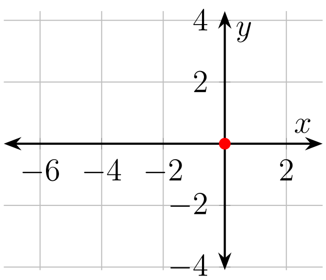


Fig. 1.2: $-4.5 < x < 4.5$

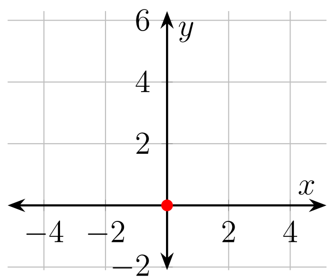


Fig. 1.3: $-2.5 < x < 6.5$

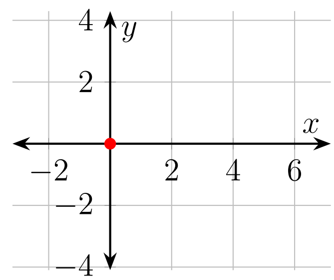
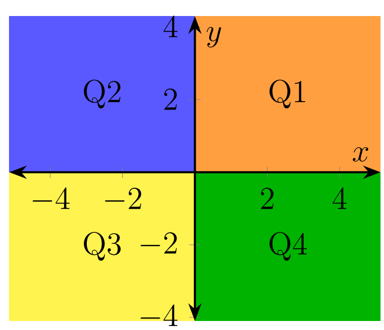


Figure 1.1, just above, makes use of the coordinate plane to show x, between -6 and 2. Then, Figure 1.2, just above, makes use of the coordinate plane to show y, between -2 and 6. Figure 1.3, just above, makes use of the coordinate plane, to show x between -2 and 6.

That the coordinate plane can be divided into 4 sections. These 4 sections are called quadrants.

- Quadrant 1 is shown in orange, labeled Q1
- Quadrant 2 is shown in blue, labeled Q2
- Quadrant 3 is shown in yellow, labeled Q3
- Quadrant 4 is shown in green, labeled Q4



Definition 1.1 — Coordinate Pair. A coordinate pair is a pair of numbers that describes the position of a point on the coordinate plane. A coordinate pair is written in parentheses. The first number is called the x-coordinate. The second number is called the y-coordinate. In the coordinate pair (3,2), the x-coordinate would be 3, and the y-coordinate would be 2.

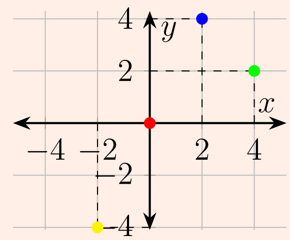
Tip A coordinate pair is also called an ordered pair.

If one is looking down at a city grid, or a forest, or a park, in the same way one could tell someone to walk forward, or backwards. One could tell someone to walk to the left, or the right. First, a starting point needs to be clear. In a coordinate plane, the starting point is the center, and it is called the origin. The origin is shown in the graph below, with a red dot.

Solved Problem 1.1 Graph the coordinate pairs (4, 2), (2, 4), (-4, -2), and (-2, -4)

- (4, 2) From the origin go 4 to the right and 2 up, shown with a green dot
- (2, 4) From the origin go 2 to the right and 4 up, shown with a blue dot
- (-2, -4) From the origin go 2 to the left and 4 down, shown with a yellow dot

Final Answer



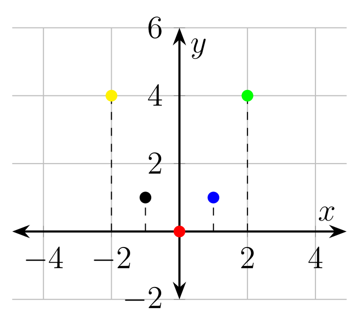
For the coordinate pair ($\boxed{4}$, 2), shown with a green dot, the x-coordinate is 4.

For the same coordinate pair (4, $\boxed{2}$), shown with a green dot, the y-coordinate is 2. ■

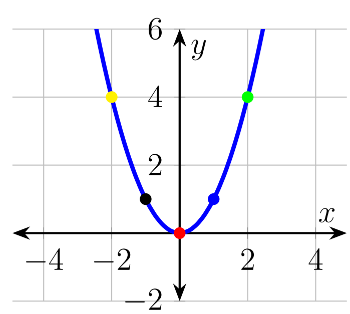
Coordinate pairs, seem like a simple concept. In an algebra course, it might be assumed that a student already knows this.

To plot coordinates (0,0) in red, from the center or origin, go 0 units to the right, and 0 units up. This is shown in the graph below with a red point. To plot coordinates (1,1) in blue, from the origin, go 1 unit to the right, and 1 unit up

- Plot coordinates (2,4) in green. From the origin, go 2 units to the right, and 4 units up
- Plot coordinates (-1,1) in black. From the origin, go 1 unit to the left, and 1 unit up
- Plot coordinates (-2,4) in yellow. From the origin, go 2 units to the left, and 4 units up
- Notice that a shape is forming, kind of like a cup



If additional points are plotted, then a curved line will form, as shown in the blue line, in the graph to the right. This blue curve, that looks kind of like a letter U or a cup, is a special curve called a **parabola**. The point here is to review graphs. A parabola will be discussed again in detail

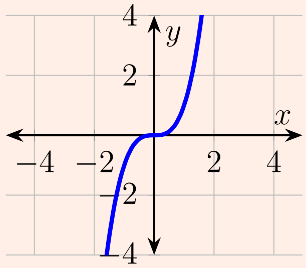


1.2 Visual Linear Functions

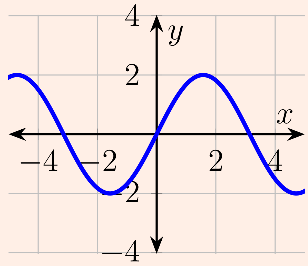
Differentiating between linear and nonlinear equations is actually easy.

Solved Problem 1.2 Which of the graphs below is the graph of a linear equation?

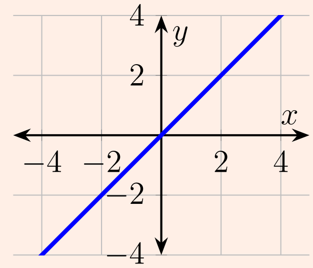
Graph A



Graph B



Graph C



Problem-solving might begin with:

- What is meant by linear?
- All 3 graphs seem to have a line?

Only Graph C has a straight line. $\boxed{\text{Final Answer} \quad \text{Graph C}}$

In the solved problem above, true all 3 graphs show a line. Above, Graph A and Graph B show lines with curves, these are referred to as nonlinear. A straight line is referred to as linear. There can be confusion in language here.

Tip Linear refers to straight lines, and nonlinear refers to curves.

Just below, to the left, one can see an expression. Just below, to the right, one can see an equation.

$$5x+3 \qquad\qquad y=5x+3$$

Tip A mathematical expression will not have an equal sign

Recall that $4^2 = 4\times 4$, $4^1 = 4$. Also, $x = x^1$. The equation $y = 2x+5$ can also be written as $y = 2x^1+5$. In the equation $y = 2x+5$, x is not squared. The exponent of x is one. In this equation the highest exponent of any variable is one. If the exponent of x is higher than one then the equation is not linear.

Definition 1.2 — Degree. The degree of a polynomial is the highest exponent of a variable that is found, in the polynomial. Linear functions have a degree of one.

Often polynomials are not linear equations and polynomials do not have to be linear equations.

In a linear equation or a linear function, any variable will have an exponent of 1 and this exponent cannot be higher than one. In the following examples one can confirm that if there is a variable, the exponent of the variable never exceeds 1.

$y = 5x + 8$ is the same as $y = 5x^1 + 8$ $\quad$ $y = 3x + 7$ is the same as $y = 3x^1 + 7$

Tip For a linear equation the exponent or degree of any variable is 1

Solved Problem 1.3 Are the following equations linear?

Is $y = x + 1/2$ linear?

$\boxed{\text{Yes, it has a degree of 1}}$

Is $y = x^3$ linear?

$\boxed{\text{No, it has a degree of 3}}$

Is $y = x^2 + 3$ linear?

$\boxed{\text{No, it has a degree of 2}}$

Is $y = x + 3^5$ linear?

$\boxed{\text{Yes, it has a degree of 1}}$

In equation $y = x + 3^5$, the variable x has an exponent of 1, so the degree is one. ■

Figure 1.4, below, is nonlinear, and Figure 1.5 is linear.

Would a business rather have sales growth like Figure 1.4 or Figure 1.5?

One can see that Figure 1.4 shows much faster growth.

Fig. 1.4: Nonlinear

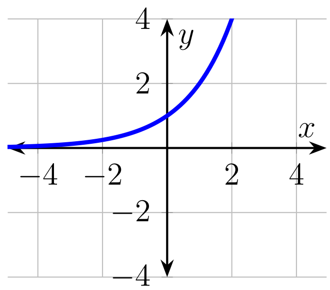
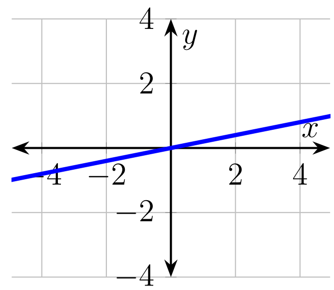


Fig. 1.5: Linear



Human populations, business growth, the revenue of a company, and bacterial growth, could be described with behavior similar to that shown in the graph in Figure 1.4. The discussion above illustrates the difference between nonlinear and linear equations.

Notice, in the 3 graphs below, that all 3 graphs have a vertical, red, line with dashes. One needs to ask, "Does the vertical red line touch the blue graph more than once?"

Definition 1.3 — Vertical Line Test. If a vertical line touches more than one point on the graph, then it is NOT a function.

Fig. 1.6: Parabola

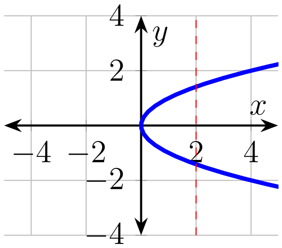


Fig. 1.7: Parabola

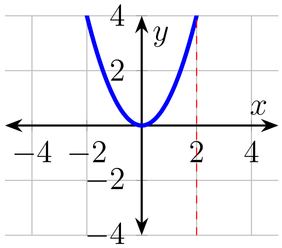
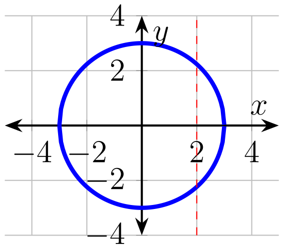


Fig. 1.8: Circle



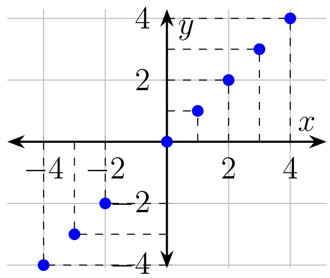
Notice in Figures 1.6 and 1.8, that the vertical, red line touches the blue graph 2 times. Now, notice that in Figure 1.7, the vertical, red line **NEVER** touches the blue graph more than once, at a time. Only Figure 1.7 shows a function. The other 2 graphs do not show functions.

Notice, below, that the 8 blue dots form a straight line. This line is linear, because it is a straight line. This line is also a function. One can add more dots, but it is already possible to see the shape of a straight line.

- Coordinate (-4, -4)
- Coordinate (-3, -3)
- Coordinate (-2, -2)
- Coordinate (0, 0)
- Coordinate (1, 1)
- Coordinate (2, 2)
- Coordinate (3, 3)
- Coordinate (4, 4)

Practice finding each coordinate pair!

Fig. 1.9: Linear Function



What if one is in a hurry? Is it necessary to plot all 8 points to describe a straight line? All that is needed to form a linear function is two points and a ruler. A linear function is just a straight line.

Coordinate pairs (-2, -2) and (2, 2) are shown as blue dots, in Figure 1.10

If one draws a line across the blue dots, a straight line is formed, as shown in Figure 1.11

Fig. 1.10: 2 Points

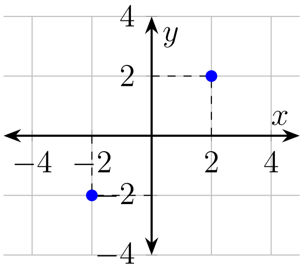
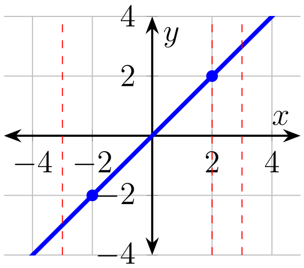


Fig. 1.11: Linear Function



Coordinate pairs (-4, -2) and (4, 2) are shown in blue dots, in Figure 1.12

If one draws a line across the blue dots, a straight line is formed, as shown in Figure 1.13

Fig. 1.12: 2 Points

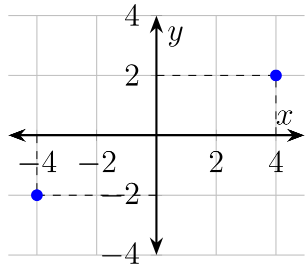
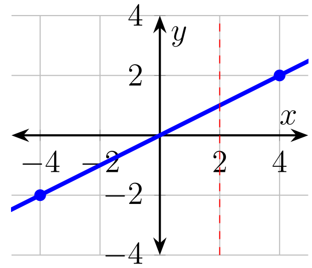


Fig. 1.13: Linear Function



In Figure 1.13, above, the vertical-line test in red, again, confirms that this is a function. This is because the vertical red line never touches the blue line more than once.

Tip All that is needed to draw a linear function is 2 points

The term "slope" is algebraic jargon, and it also describes a rate of change. Still, slope is really just a way of describing a hill, slant, or inclination .

Fig. 1.14: Zero Slope

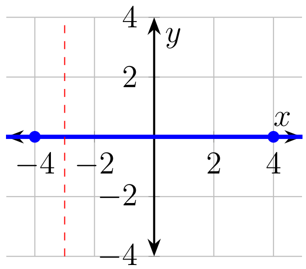


Fig. 1.15: "Some" Slope

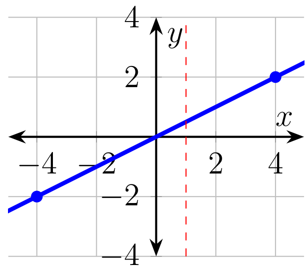
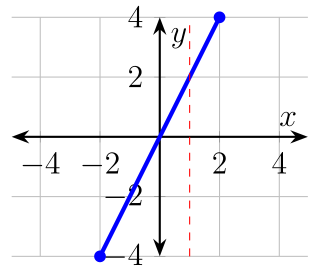


Fig. 1.16: "Greater" Slope



In Figure 1.14, Figure 1.15, and Figure 1.16, three linear functions are shown. All three graphs are functions and straight lines. The graph in Figure 1.16 has a greater slope than the graph in Figure 1.15. **The graph in Figure 1.14 is said to have a slope of zero**. A slope of zero is just a horizontal line!

Tip Visual cues or characteristics of linear functions are that they are straight, and they pass the vertical-line test

The coordinate pair (0, 2) is shown with a blue dot, in Figure 1.17

The slope is zero

Fig. 1.17: Step 1 Point

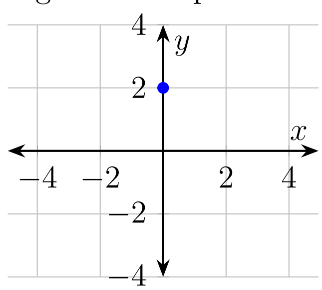
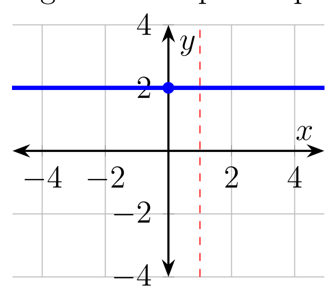


Fig. 1.18: Step 2 Slope



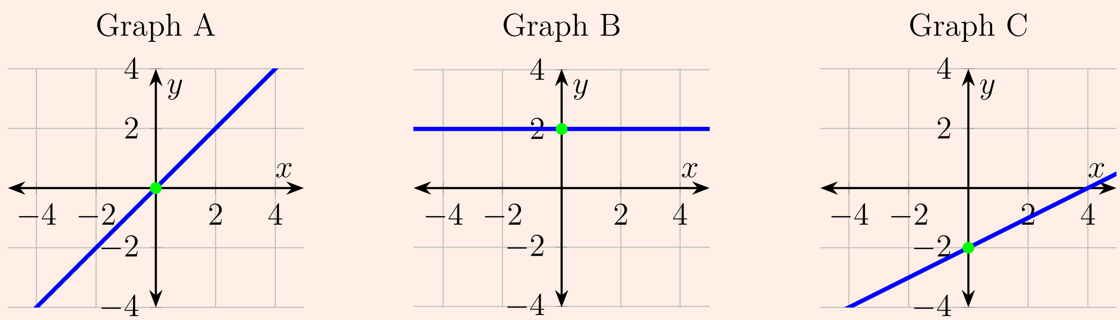
If one takes a ruler and draws a horizontal, straight line across the blue dot in the graph in

Figure 1.17, then one has formed a linear function with a slope of zero.

Notice that in the graph in Figure 1.18 the blue line intersects the y-axis at the point (0,2). The graph in Figure 1.18 does clearly touch the y-axis, and this occurs at positive 2 on the y-axis. Also, for any point on the y-axis the x-coordinate is always zero, so for the graph in Figure 1.18, the y-intercept is simply 2.

Definition 1.4 — Y-Intercept. A y-intercept is the point at which a line crosses the y-axis. In other words a y-intercept is the point at which a line intersects the y-axis. At a y-intercept the x-coordinate is zero.

Solved Problem 1.4 Find the y-intercept for the following graphs



In all three graphs above, the y-intercept is shown with a green dot.

Final Answer For Graph A the y-intercept is simply zero or point (0,0)

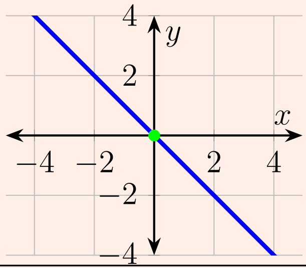
Final Answer For Graph B the y-intercept is simply 2 or point (0,2)

Final Answer For Graph C the y-intercept is simply -2 or point (0,-2)

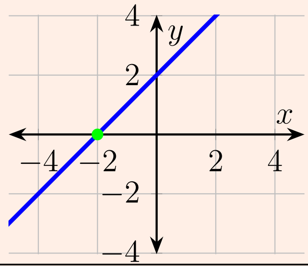
Definition 1.5 — X-Intercept. An x-intercept is the point at which a line crosses the x-axis. In other words an x-intercept is the point at which a line intersects the x-axis. At an x-intercept the y-coordinate is zero.

Solved Problem 1.5 Find the x-intercept for the following graphs

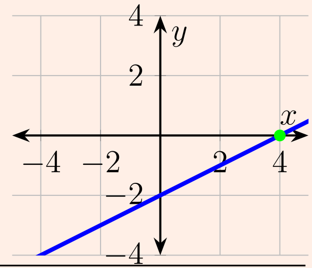
Graph D



Graph E



Graph F



In all three graphs above, the x-intercept is shown with a green dot.

Final Answer For Graph D the x-intercept is simply zero or point (0,0)

Final Answer For Graph E the x-intercept is simply -2 or point (-2,0)

Final Answer For Graph F the x-intercept is simply 4 or point (4,0) ▪

With an understanding of y-intercept let's describe linear functions again. Any point and the slope will be enough to describe a straight line. The x-intercept and the slope could be used to describe a line. In algebra, it is common to describe a line with the y-intercept and the slope.

The y-intercept is -3 shown with a blue dot, in Figure 1.19

The slope is zero

Fig. 1.19: Step 1 Point

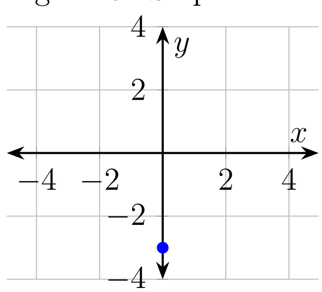
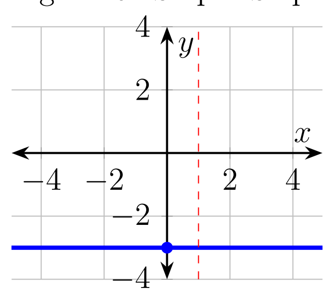


Fig. 1.20: Step 2 Slope



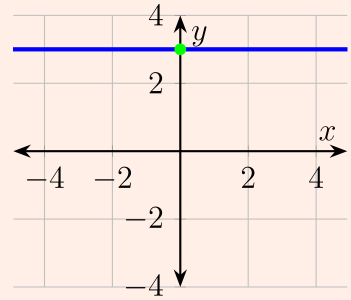
Recall, that a zero slope means a horizontal line. A horizontal line is drawn across the point (0, -3) to form a linear function with slope zero.

Solved Problem 1.6 Graph a linear function with a y-intercept of 3, and a slope of zero.

Problem-solving begins with thinking in terms of components:

- How do I plot the y-intercept?
- What does a slope of zero look like?
- A y-intercept of 3 is the same as the coordinate pair (0, 3)

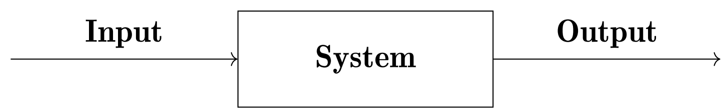
Final Answer



In the graph above, the y-intercept of 3 is shown with a green dot. Then all that is needed is to draw a horizontal straight line across the green dot.

A vertical line test can be understood in a very visual way. At this point, it is a good time to establish a deeper understanding of a function. This can still be done in a very simple and visual way. Why is a function called a function?

Fig. 1.21: A Simple System



A function can be thought of as a system. A system can be a very simple concept that has one input, a box, and one output. This is shown in the diagram above.

A system is all about focusing on the useful information and ignoring the rest. How so? There are many simple examples of a system. A coffee maker is quite simple, and a coffee maker can be thought of as a system. One can choose to focus on the coffee grounds and water that go into the coffee maker and the hot coffee that comes out of the coffee maker. It can then be said that a coffee maker has 2 inputs and one output. This is shown in the simple diagram below.

Fig. 1.22: A Coffee Maker as a System

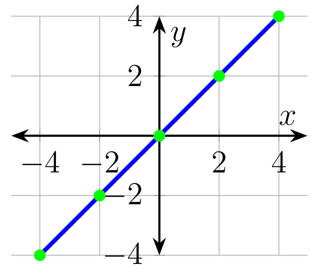
The coffee maker needs electricity, it may have a microchip inside, it is made of plastic, but all this other information, is unnecessary. When problem-solving one needs to focus on key information. So, the simple diagram above only considers the coffee grounds and water that go in, and the hot coffee that comes out.

A coordinate pair has an input and an output, and a coordinate pair reflects that a function is like a system. In a coordinate pair the x-coordinate is the input, and the y-coordinate is the output.

In the following coordinate pair, each number is surrounded by an orange border ($\boxed{5}$, $\boxed{2}$). The first number 5, is the x-coordinate, and it is the input. The second number 2, is the y-coordinate, and it is the output. Coordinate pairs have an input, output nature. The graph in Figure 1.23, below, to the right, shows a linear function in blue. On this blue line, coordinate pairs are shown in green.

- If the x-coordinate is -4, the y-coordinate is -4
- If the x-coordinate is -2, the y-coordinate is -2
- If the x-coordinate is 2, the y-coordinate is 2
- If the x-coordinate is 4, the y-coordinate is 4

Fig. 1.23: Input - Output

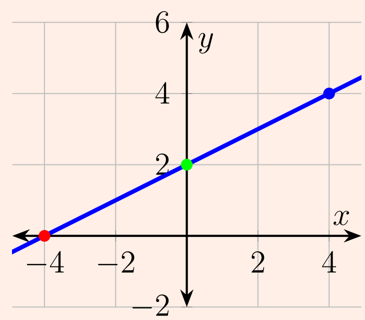


For every coordinate pair (x, y), shown in green, above, an x is associated to a y. For a linear function, the x-coordinate is the input, and the y coordinate is the output. If one has an x-coordinate, one can observe the graph in Figure 1.23, in order to find the y-coordinate.

Solved Problem 1.7 For the graph below, find the y-coordinate outputs that correspond to the x-coordinate inputs -4, 0, and 4

Think in terms of components:

- X-coordinates are provided
- A linear function is also provided
- One needs to find y-coordinates
- In every coordinate pair the x-coordinate is the input and the y-coordinate is the output



The coordinate pair for the point (-4, $\boxed{0}$), is shown in red. The corresponding y-coordinate is zero.

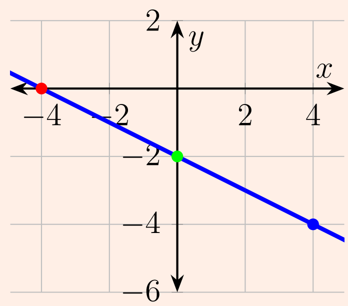
The coordinate pair for the point (0, $\boxed{2}$), is shown in green. The corresponding y-coordinate is 2.

The coordinate pair for the point (4, $\boxed{4}$), is shown in blue. The corresponding y-coordinate is 4.

Final Answers If x is -4, y is 0 If x is 0, y is 2 If x is 4, y is 4

Solved Problem 1.8 For the graph below find the y-coordinate outputs that correspond to the x-coordinate inputs -4, 0, and 4.

- For the red point, what is the corresponding y-coordinate?
- For the green point, what is the corresponding y-coordinate?
- For the blue point, what is the corresponding y-coordinate?



In the graph above, the 3 points of interest are shown in red, green, and blue

Final Answers If x is -4, y is 0 If x is 0, y is -2 If x is 4, y is -4

Fig. 1.24: y as a Function of x

Input x → $y =$ **function of** x / $y = f(x)$ / y **of** x → **Output** y

In other words for every point on the function, x is an input, and y is an output. This is why in algebra one writes, y as a function of x, $y = f(x)$, or y of x. Understanding that a function can operate as a simple system with one input and one output is written in the following ways.

- y as a function of x
- $y = f(x)$
- y of x

1.3 Solving for an Unknown

Reverse-PEMDAS, can be helpful while learning to solve for an unknown.

S - Subtraction
A - Addition
D - Division
M - Multiplication
E - Exponents
P - Parentheses

What is reverse-PEMDAS ? Simply, reverse the PEMDAS acronym.

Below we will demonstrate how to use reverse-PEMDAS to solve a simple equation.

$3x+4=7$ The variable x is unknown

The first step would be to find any subtraction and eliminate it. The next step is to find any addition and eliminate it. This expression does have addition, and in order to eliminate addition, one subtracts 4, from both sides. This is shown below.

$3x+4\ \boxed{\text{-4}} = 7\ \boxed{\text{-4}}$ this leads to $3x=3$

Notice, just above, how $+4$ is eliminated by subtracting 4, from both sides. The next step is to find any division and eliminate it. The expression $3x=6$, does not have any division. The next step is to find any multiplication, and eliminate it. In order to eliminate multiplication, one needs to divide, both sides by 3. This is demonstrated below.

$\frac{3x}{3}=\frac{3}{3}$ leads to $\frac{\cancel{3}x}{\cancel{3}}=\frac{3}{3}$ then $\boxed{x=1}$ Recall that $\frac{3}{3}=1$

The steps below, demonstrate how to solve for the value of the unknown variable x, given the equation $3x-4=-1$.

$3x-4=-1$ $3x-4+4=-1+4$ $3x=3$

Below is an example of how we eliminate multiplication by division.

$3x=3$ $\frac{3x}{3}=\frac{3}{3}$ $\frac{\cancel{3}x}{\cancel{3}}=\frac{\cancel{(3)}1}{\cancel{(3)}1}$ $\boxed{x=1}$

Solving for an unknown is demonstrated again, below.

$4x-2=6$ $4x-2+2=6+2$ $4x=8$

$4x=8$ $\frac{4x}{4}=\frac{8}{4}$ $\frac{\cancel{4}x}{\cancel{4}}=\frac{\cancel{(4)}2}{\cancel{(4)}}$ $\boxed{x=2}$

It can be helpful for a student to think of reverse-PEMDAS in the following way.

S - Subtraction	If subtraction exists, then eliminate it.
A - Addition	If addition exists, then eliminate it.
D - Division	If division exists, then eliminate it.
M - Multiplication	If multiplication exists, then eliminate it.
E - Exponents	If an exponent exists, then eliminate it.
P - Parentheses	If parentheses exist, then eliminate them.

Solved Problem 1.9 Solve for x in the following linear equations:

(a) $5x-8=7$ (b) $2x-5=5$

(a) $5x-8=7$ First, if subtraction exists eliminate it. Subtraction is eliminated by adding 8 to both sides. This is shown below.

$$5x-8+8=7+8 \quad 5x=15 \quad \frac{5x}{5}=\frac{15}{5} \quad \frac{\cancel{5}x}{\cancel{5}}=\frac{\cancel{(5)}3}{\cancel{(5)}} \quad \boxed{x=3}$$

In this equation, addition does not exist, so this step is skipped. Division does not exist, so this step is skipped. Multiplication exists, and it is eliminated by dividing both sides of the equation, by 5. This is shown above.

After multiplication, an exponent does not exist, and parentheses do not exist. More importantly the unknown variable x has been isolated, and it is equal to 3.

(b) $2x-5=5$ The first step is, if subtraction exists eliminate it. Subtraction is eliminated by adding 5 to both sides of the equation. Then, addition and division do not exist. The fourth step is if multiplication exists, then eliminate it. Multiplication is eliminated by dividing both sides of the equation by 2. This is shown, below.

$$2x-5=5 \quad 2x-5+5=5+5 \quad \frac{2x}{2}=\frac{10}{2} \quad \frac{\cancel{2}x}{\cancel{2}}=\frac{\cancel{(2)}5}{\cancel{(2)}} \quad \boxed{x=5}$$

■

In many cases, there are two ways to think about solving the equation.

$\frac{4x}{2}=14$ Eliminate division

$$2\left(\frac{4x}{2}\right)=2(14) \qquad \cancel{2}\left(\frac{4x}{\cancel{2}}\right)=28$$

$4x=28$ Eliminate multiplication

$$\frac{4x}{4}=\frac{4(7)}{4} \qquad \frac{\cancel{4}x}{\cancel{4}}=\frac{\cancel{4}(7)}{\cancel{4}} \qquad \boxed{x=7}$$

$\frac{4x}{2}=14$ Simplify the fraction

$$\frac{2(2x)}{2}=14 \qquad \frac{\cancel{2}(2x)}{\cancel{2}}=14$$

$2x=14$ Eliminate multiplication

$$\frac{2x}{2}=\frac{2(7)}{2} \qquad \frac{\cancel{2}x}{\cancel{2}}=\frac{\cancel{2}(7)}{\cancel{2}} \qquad \boxed{x=7}$$

The column on the left, just above, makes use of reverse-PEMDAS. The column on the right is a simpler route, if one can see that the fraction simplifies. With practice, it should be obvious that $(4x)/2$ is equal to $2x$, so the column on the right would be the best route. The column on the left is an introductory reverse-PEMDAS route.

In the right column, just above, the fraction on the left side of the equal sign is simplified. When a fraction is simplified in this way, one has not changed the left side of the equation.

Solved Problem 1.10 Solve for x in the following linear equation: $\dfrac{10x}{5} - 4 = 4$

First if subtraction exists eliminate it. Subtraction is eliminated by adding 4 to both sides of the equation. This is shown below.

$$\frac{10x}{5} - 4 \boxed{+4} = 4 \boxed{+4} \qquad \frac{10x}{5} = 8$$

The next step is, if addition exists, eliminate it. Addition does not exist. Next, if division exists eliminate it. Multiplication eliminates division, but the multiplication must be applied to both sides, as shown below.

$$5\left(\frac{10x}{5}\right) = 5(8) \qquad \cancel{5}\left(\frac{10x}{\cancel{5}}\right) = 40 \qquad 10x = 40$$

Next, if multiplication exists, eliminate it. Both sides can be divided by 10, in order to eliminate 10, and isolate the variable x. This is shown below.

$$\frac{10x}{10} = \frac{40}{10} \qquad \frac{\cancel{10}\,(x)}{10} = \frac{\cancel{10}\,(4)}{\cancel{10}} \qquad \boxed{x = 4}$$

This last solved problem demonstrated the use of reverse-PEMDAS. This alternative route is demonstrated below.

$$\frac{10x}{5} = 8 \qquad \frac{\cancel{5}(2x)}{\cancel{5}} = 8 \qquad 2x = 8 \qquad \frac{\cancel{2}x}{\cancel{2}} = \frac{8}{2} \qquad \boxed{x = 4}$$

Just above, the equation $2x = 8$ does not require reverse-PEMDAS. Division can be used to eliminate multiplication. For this reason, both sides are divided by 2, as shown above. When solving for an unknown it can also be useful to be able to combine like terms. This is demonstrated below.

$$\boxed{\text{8x}} + \boxed{\text{2x}} - \boxed{\text{3x}} - 7 = 0 \qquad 7x - 7 = 0$$

These 3 terms would be considered like terms. These three terms are called like terms because they have the same variable. For a linear equation the degree is 1. The three like terms can be added and subtracted together, so that they become one term. This means that the coefficients 8, 2, and -3 can simply be added, which leads to a coefficient of 7, in the term 7x. It is about the same as realizing that adding 8 potatoes plus 2 potatoes minus 3 potatoes is equal to 7 potatoes.

The new piece of understanding is deciding which terms are like terms. Like terms have the same variable and this variable has the same exponent.

$$x^1 = x \qquad 8x^1 = 8x \qquad 2x^1 = 2x \qquad 3x^1 = 3x$$

Solved Problem 1.11 Combine like terms in the following equations.

(a) $-4x + 6x + 10x - 12 = 0$ (b) $2x + 3x + 7y + 3y - 15 = 0$

(a) $\boxed{-4x} + \boxed{\text{6x}} + \boxed{\text{10x}} - 12 = 0$ Three like terms are shown here with an orange border. In order to combine them one simply adds up the coefficients, and this result is combined with the same variable. So -4+6+10=12, which leads to the new simpler term 12x. The simplified equation is then $\boxed{12x - 12 = 0}$.

(b) Recall that:

- Like terms have the same variable.
- For like terms, variables have the same exponent.

This equation has 2 different variables x and y. These 2 variables are then handled separately. The 2 like terms with the variable x are shown here with an orange border, $\boxed{\text{2x}} + \boxed{\text{3x}} + 7y + 3y - 15 = 0$. The coefficients 2 and 3 are added, 2+3=5, so these 2 terms become $5x$. The equation becomes $5x + 7y + 3y - 15 = 0$. Next, the 2 terms with the y variable are shown with an orange border $5x + \boxed{\text{7y}} + \boxed{\text{3y}} - 15 = 0$. The coefficients 7 and 3 are added, 7+3=10, so these 2 terms become $10y$. The simplified equation becomes $\boxed{5x + 10y - 15 = 0}$. ■

Combining like terms with linear equations can be quite simple. The next chapter includes working with an exponent of 2. Solving for an unknown, and combining like terms is revisited in the next chapter.

1.4 Algebra of Linear Functions

Describing a line involves both describing its shape and its location.

The previous section, in this chapter discussed in a visual way, how a straight line can be described or defined.

- A straight line can be described with 2 points
- Also, a straight line can be described with only 1 point and a slope

If a straight line can also be described with a point and a slope. It is necessary to fully describe the slope of a line. Slope is a rate of change. In a coordinate plane, slope describes how y changes with respect to x. All 3 slope equations below describe this. Intuitively, slope describes steepness or inclination as one might find on any hill or ski slope.

Definition 1.6 — Slope. Slope is simply a measure of the steepness or inclination of a line.

$$\text{Slope} = \frac{\text{rise}}{\text{run}} \qquad \text{Slope} = \frac{\text{vertical change}}{\text{horizontal change}} \qquad \text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

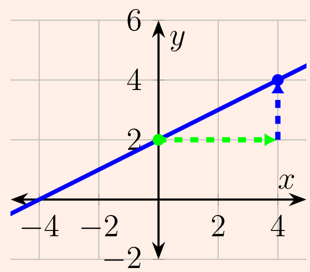
Slope is also said to be a rate of change. For a rate of change one quantity changes with respect to another. Examples of this are miles per hour and kilometers per hour.

Solved Problem 1.12 Find the slope of the blue line in the graph below.

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

In the graph, a blue, dashed line shows the rise, which is $\boxed{2}$ units.

In the graph, a green, dashed line shows the run, which is $\boxed{4}$ units.



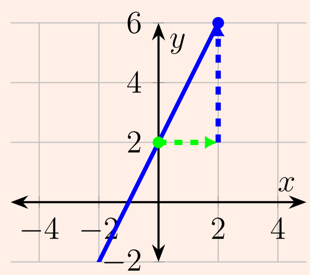
Once the rise and run have been identified they are simply plugged in to the following equation. $\text{slope} = \frac{\boxed{\text{rise}}}{\boxed{\text{run}}} = \frac{2}{4} = \frac{1}{2}$ $\boxed{\text{Final Answer} \quad \text{The slope is } \frac{1}{2}}$ ■

Solved Problem 1.13 Find the slope of the blue line in the graph below.

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

A blue, dashed line shows the rise, which is $\boxed{4}$ units.

A green, dashed line shows the run, which is $\boxed{2}$ units.



$$\text{slope} = \frac{\boxed{\text{rise}}}{\boxed{\text{run}}} = \frac{4}{2} = 2$$

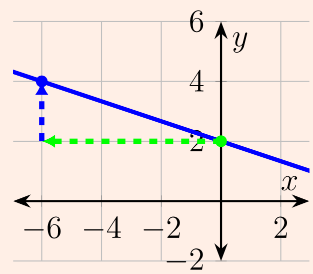
$\boxed{\text{Final Answer} \quad \text{The slope is } 2}$ ■

Solved Problem 1.14 Find the slope of the blue line in the graph below.

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

A blue, dashed line shows the rise, which is $\boxed{2}$ units.

A green, dashed line shows the run, which is $\boxed{-6}$ units.



$$\text{slope} = \frac{\boxed{\text{rise}}}{\boxed{\text{run}}} = \frac{2}{-6} = \frac{1}{-3}$$

$\boxed{\text{Final Answer} \quad \text{The slope is } -\frac{1}{3}}$

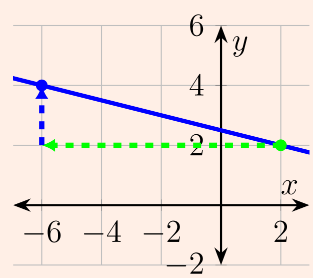
Notice that this time the run goes to the left of the y-axis. This makes the run a negative value. This makes the slope a negative value. ▪

Solved Problem 1.15 Find the slope of the blue line in the graph below.

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

A blue, dashed line shows the rise, which is $\boxed{2}$ units.

A green, dashed line shows the run, which is $\boxed{\text{-8}}$ units.



$$\text{slope} = \frac{\boxed{\text{rise}}}{\boxed{\text{run}}} = \frac{2}{-8} = \frac{1}{-4}$$

$\boxed{\text{Final Answer} \quad \text{The slope is } -\frac{1}{4}}$ ▪

On a ski slope or on any hill, one is either going up the hill or down the hill. For this reason, height or position is changing.

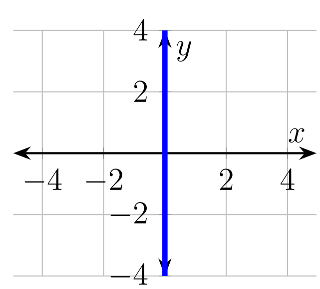
What is the slope of a vertical line? For a vertical line the rise would seem to rise forever. The run would be zero. If one divides by zero the result is said to be undefined.

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

It seems to rise forever

The run would be zero

$$\text{slope} = \frac{\boxed{\text{rise}}}{\boxed{\text{run}}} = \frac{\text{rise}}{0} = \text{undefined}$$



Tip The slope of a horizontal, straight line is zero

Tip The slope of a vertical, straight line is **undefined**

Three formulas for slope have been shown. They have the same meaning.

$$\text{slope} = \frac{\text{rise}}{\text{run}} \qquad \text{slope} = \frac{\text{vertical change}}{\text{horizontal change}} \qquad \text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

Vertical change is the same as rise that was illustrated in the last 4 solved problems. Horizontal change is the same as run that was illustrated in the last 4 solved problems. The third slope formula just above also has the same meaning, but it is worthwhile to apply it. This third slope formula can be very helpful when describing a line.

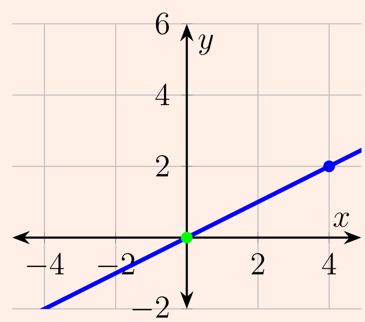
Solved Problem 1.16 Find the slope of the blue line in the graph below.

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

One has to select 2 points, $(x_2,\ y_2)$ and $(x_1,\ y_1)$

Point 2, is the blue dot (4, 2)

Point 1, is the green dot (0, 0)



For point 2, the blue dot, the coordinates are (4, 2), so $x_2 = 4$ and $y_2 = 2$
For point 1, the green dot, the coordinates are (0, 0), so $x_1 = 0$ and $y_1 = 0$

$$\text{slope} = \frac{\boxed{y_2 - y_1}}{\boxed{x_2 - x_1}} = \frac{2-0}{4-0} = \frac{2}{4} = \frac{1}{2}$$

Final Answer The slope is $\frac{1}{2}$

A visual approach is great, but it is necessary to be able to find the slope without a graph.

Solved Problem 1.17 Find the slope of a line that has the two points (4, 6) and (0, 0)

A graph has not been provided, but there is a formula for slope that only requires the coordinates for 2 points. $\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$

Two points have been provided in the form of coordinate pairs, $(x_2,\ y_2)$ and $(x_1,\ y_1)$

Point 2, $(x_2,\ y_2)$ can be the point (4, 6), so $\boxed{x_2 = 4}$ and $\boxed{y_2 = 6}$

Point 1, $(x_1,\ y_1)$ can be the point (0, 0), so $\boxed{x_1 = 0}$ and $\boxed{y_1 = 0}$

Once the necessary coordinates have been identified, then they are just plugged into the formula below.

$$\text{slope} = \frac{\boxed{y_2 - y_1}}{\boxed{x_2 - x_1}} = \frac{6-0}{4-0} = \frac{6}{4} = \frac{3}{2} \qquad \boxed{\text{Final Answer} \quad \text{The slope is } \frac{3}{2}}$$

■

Solved Problem 1.18 Find the slope of a line that has the two points (-2, 6) and (-4, 0)

A graph has not been provided, but there is a formula for slope that only requires the coordinates for 2 points. $\text{slope} = \dfrac{y_2 - y_1}{x_2 - x_1}$

Two points have been provided in the form of coordinate pairs, $(x_2,\ y_2)$ and $(x_1,\ y_1)$

Point 2, $(x_2,\ y_2)$ can be the point (-2, 6), so $\boxed{x_2 = -2}$ and $\boxed{y_2 = 6}$

Point 1, $(x_1,\ y_1)$ can be the point (0, 0), so $\boxed{x_1 = -4}$ and $\boxed{y_1 = 0}$

Once the necessary coordinates have been identified, then they are just plugged into the formula below. Recall that $-2-(-4) = -2+4$

$$\text{slope} = \frac{\boxed{y_2 - y_1}}{\boxed{x_2 - x_1}} = \frac{6-0}{-2-(-4)} = \frac{6-0}{-2+4} = \frac{6}{2} = 3 \qquad \boxed{\text{Final Answer} \quad \text{The slope is } 3}$$

■

It was previously mentioned that a straight line or linear function can be described if one has a point on the line, and it's slope. This was explained, demonstrated, and applied in a previous section. Feel free to review Chapter 1.2: Visual Linear Functions.

Definition 1.7 — Point-Slope Form. In point-slope form, a line is described by the following equation, where the point (x_1, y_1) is any point on the line

$$y - y_1 = m(x - x_1)$$

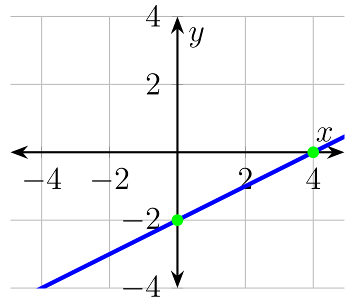
m = The slope of the line

x_1 is the x-coordinate and y_1 is the y-coordinate

2 points shown in green are selected

$$\text{slope} = \frac{\boxed{y_2 - y_1}}{\boxed{x_2 - x_1}}$$

$$\text{slope} = \frac{0-(-2)}{4-0} = \frac{0+2}{4} = \boxed{\frac{1}{2} = m}$$



Once the slope is found, as shown above, it becomes possible to describe the line in point-slope form. The slope $m = \frac{1}{2}$. For the point (4, 0), $y_1 = 0$, and $x_1 = 4$. This is just plugged into the point-slope form equation, as shown below.

$$y - y_1 = m(x - x_1) \;=\; y - \boxed{y_1} = \boxed{m}(x - \boxed{x_1}) \;=\; y - 0 = \frac{1}{2}(x-4)$$

Notice, just above, that the variables with subscripts, y_1 and x_1 are surrounded by an orange border. This is where the coordinates go. The other two variables y and x are left as is.

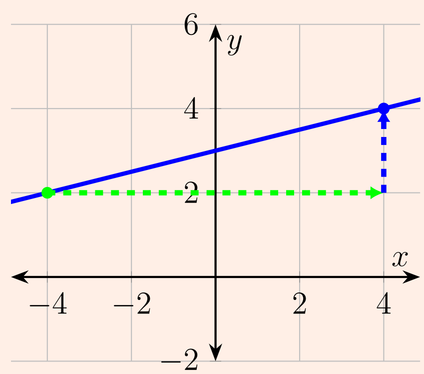
Solved Problem 1.19 Find the point-slope form of the line in the graph below

To find the slope, two points are selected.

$point_2 = (4,4)$ and $point_1 = (-4,2)$

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{slope} = \frac{4-2}{4-(-4)} = \frac{2}{4+4} = \frac{2}{8} = \frac{1}{4}$$



Once the slope has been calculated, the line can be described in point-slope form. The slope $m = \frac{1}{4}$.

Then, either of the 2 points can be used in the point-slope form equation. For the point (-4, 2), $x_1 = -4$ and $y_1 = 2$. This is just plugged into the point-slope form equation, as shown below. Recall that $x - (-4) = x + 4$

$$y - y_1 = m(x - x_1) \;=\; y - \boxed{} = \boxed{}(x - \boxed{}) \;=\; y - 2 = \frac{1}{4}(x - (-4))$$

$$\boxed{\text{Final Answer} \qquad y - 2 = \frac{1}{4}(x+4)}$$

■

Solved Problem 1.20 Given that a straight line has the points (2, 4) and (-4, 0), find the point-slope form of the line

---

Two points have been provided $point_2 = (2,4)$ and $point_1 = (-4,0)$

slope $= \dfrac{y_2 - y_1}{x_2 - x_1}$ slope $= \dfrac{4-0}{2-(-4)} = \dfrac{4}{2+4} = \dfrac{4}{6} = \dfrac{2}{3}$

Once the slope has been calculated, the line can be described in point-slope form. The slope $m = \frac{2}{3}$. Recall, that either of the two points can be used in the point-slope form equation.

For the point (2,4), $x_1 = 2$ and $y_1 = 4$. This is plugged into the point-slope form equation.
$y - y_1 = m(x - x_1) \;=\; y - \boxed{} = \boxed{}(x - \boxed{}) \;=\; y - 4 = \frac{2}{3}(x-2)$

Final Answer $y - 4 = \dfrac{2}{3}(x-2)$

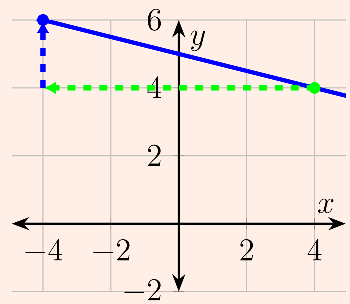
Solved Problem 1.21 Graph the line described by $y - 4 = -\frac{1}{4}(x-4)$

The equation is in point-slope form $y - y_1 = m(x - x_1)$

In $y - \boxed{4} = -\frac{1}{4}(x - \boxed{4})$ the point is (4, 4). This point is shown on the graph in green.

The slope is $-\frac{1}{4}$, so rise=2, run=-8

Final Answer



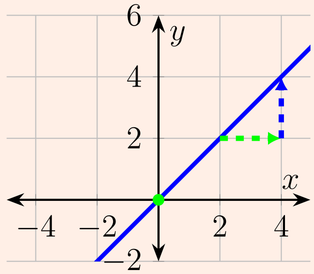
Solved Problem 1.22 Graph the line described by the equation $y = x$

Is the equation above in point-slope form? $y - y_1 = m(x - x_1)$

What is the slope? Is there a point in the initial equation $y = x$?

Is there hidden information in the initial equation?

Final Answer



Recall that $1 - 0 = 1$, $y - 0 = y$, $x - 0 = x$, $y = 1 \times y$, $x = 1 \times x$, $\frac{1}{1} = 1$

So, the initial equation $y = x$ can be rewritten as $y - 0 = 1(x - 0)$

The equation $y - \boxed{0} = 1(x - \boxed{0})$ does have a hidden point of (0,0), and this is shown above as a green dot. This equation $y - 0 = \boxed{1}(x - 0)$ also has a slope of $\frac{rise}{run} = 1$. ▪

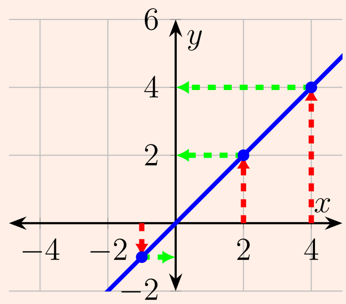
It could be said that this last solved problem is quite easy. With a little effort, one can confirm that $y = x$ is quite basic, and it will be possible to recognize $y = x$ right away.

Solved Problem 1.23 For the linear function, $y = x$, shown below, find the outputs that correspond to the inputs -1, 2, and 4.

Input x

If x_1 = -1
If x_2 = 2
If x_3 = 4

Function



Output y

y_1 = ?
y_2 = ?
y_3 = ?

The output y_1 would be the output that corresponds to x_1. Likewise, The output y_2 would be the output that corresponds to x_2. The inputs x_1, x_2, x$_3$ are simply different values of x.

For the input x_2, one can start at the point (2, 0) or more simply $x = 2$, and then one follows the red, dashed arrow to the blue point on the blue line. From there one follows the green, dashed arrow to the left, to the point (0, 2), or more simply $y = 2$. So the input x_2 corresponds to the output y_2 of 2.

- Notice that for $y = x$ the output is always equal to the input

- Notice the diagonal inclination for $y = x$, and $\boxed{\text{the slope is equal to 1}}$
- $y = x$ can also be written $y - 0 = 1(x - 0)$

$\boxed{\text{Final Answer} \qquad y_1 = -1 \quad y_2 = 2 \quad y_3 = 4}$ ■

Point-slope form recap, $y - y_1 = m(x - x_1)$:

- m is the slope
- It makes use of both the x and y coordinates of one coordinate pair
- x_1 is the x-coordinate of one coordinate pair
- y_1 is the y-coordinate of one coordinate pair
- All that is necessary is the slope and any point on the line
- If the slope is unknown then 2 points on the line are needed to find the slope

Once a few minutes are invested in understanding $y = x$ it is possible to understand what happens if $y = x$ is shifted up. If $y = x$ is shifted up 2 units then one writes $y = x + 2$. If $y = x$ is shifted up 4 units then one writes $y = x + 4$. This is shown clearly below.

Fig. 1.25: $y = x$

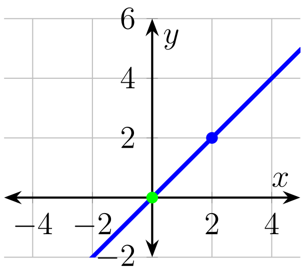


Fig. 1.26: $y = x + 2$

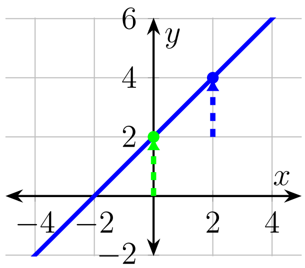
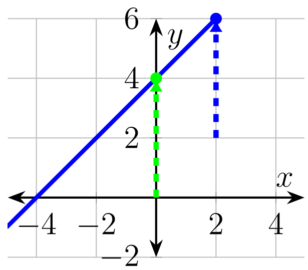


Fig. 1.27: $y = x + 4$



Just above, in Figure 1.25, one can see the graph of $y = x$, and this blue line clearly shows a blue dot and a green dot. It is straightforward to see that in Figure 1.26 and Figure 1.27, the straight blue line is simply being moved up. Figure 1.26 emphasizes this by showing that the green dot moves up 2 units. The blue dot also moves up 2 units.

Definition 1.8 — Slope-Intercept Form. In slope-intercept form, a line is described by the following equation.

$y = mx + c$	m is the slope of the line	c is the y-intercept of the line

In equation $y = x + \boxed{2}$, the y-intercept is 2. In equation $y = x + \boxed{4}$, the y-intercept is 4.

Solved Problem 1.24 Graph the following functions $y = x + 1$, $y = x + 3$, and $y = x - 1$

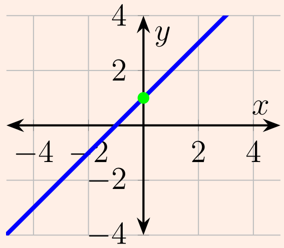
For the equation $y = x + 1$, the first step is to recognize that $y = x + 1$ is in slope-intercept form. The next step is to find the y-intercept in $y = x + \boxed{1}$, which is 1. This is shown in the graph below with a green dot. Then the equation $y = x + 1$ can be rewritten as $y = \boxed{1}\,x + 1$, so the slope is 1. A simpler solution is to recall the graph of $y = x$, that was practiced above, and just shift it up one unit!

For the equation $y = x + \boxed{3}$, the y-intercept is 3, and the slope is 1. A simple solution is to recall the graph of $y = x$, and just shift it up 3 units!

For the equation $y = x + \boxed{-1}$, the y-intercept is -1, and the slope is 1. A simpler solution is to recall the graph of $y = x$, and just shift it down one unit!

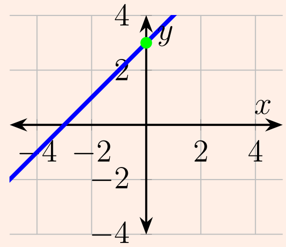
$y = x + 1$

Final Answer



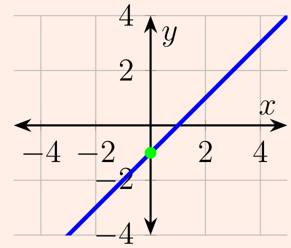
$y = x + 3$

Final Answer



$y = x - 1$

Final Answer



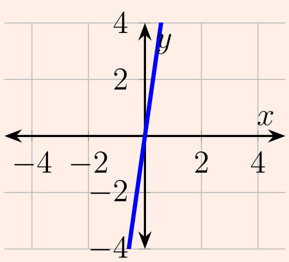
Only the y-intercept changes for the 3 graphs above. ■

Slope-intercept form may also be written as $y = mx + b$, and this will be seen in some books. The b can be any letter or symbol. The letter c is used in this book because the y-intercept is a constant.

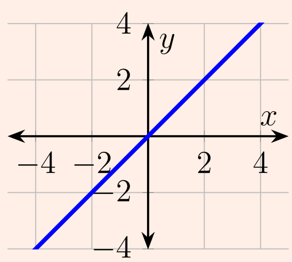
Solved Problem 1.25 Match the graphs below to the slope-intercept form for equations

$$y = \frac{1}{7}x \qquad y = x \qquad y = 7x$$

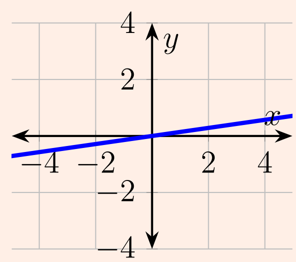
Graph A



Graph B



Graph C



This exercise involves understanding slope-intercept form and specifically slope. One must first recognize the graph of the basic equation $y = x$. Graph B is clearly the graph of $y = x$.

Then one must see that if the incline, or slope of the blue line increases, then the slope must be higher. In slope-intercept form $y = mx + c$, m will be higher. Only one of the 3 initial equations above has a slope higher than one. Also, only one of the 3 graphs above has a slope higher than one. Graph A, has a higher slope or inclination than Graph B

Only one of the 3 initial equations above has a slope lower than one. Also, only one of the 3 graphs above has a slope lower than one. Graph C, clearly has a lower inclination or slope than Graph B.

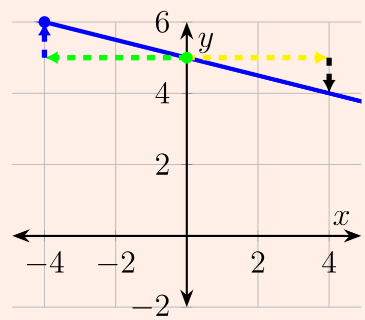
$\boxed{\text{Final Answers} \quad \text{Graph A is } y = 7x \quad \text{Graph B is } y = x \quad \text{Graph C is } y = \frac{1}{7}x}$ ■

Solved Problem 1.26 Graph the following equation $y = -\frac{1}{4}x + 5$

In the equation $y = -\frac{1}{4}x + \boxed{5}$ the y-intercept is 5. This is shown as a green dot on the graph

In the equation $y = \boxed{-\frac{1}{4}}\, x + 5$ the slope is $-\frac{1}{4}$

Final Answer



The linear function or the straight line must satisfy a slope of $-\frac{1}{4}$. This means it must have a rise of one unit and this is shown above with a blue, dashed line. The slope must have a horizontal run of -4, or 4 units to the left.

A negative slope can be handled in 2 ways. If there is a negative slope, either the rise OR the run must be negative. For the green and blue, dashed arrows, above the rise is positive, and the run is negative. For, the yellow and black, dashed arrows, above, the horizontal run is positive in yellow, and the rise is negative or downwards, in black. Either route is correct. ■

It may be the case that the y-intercept of a line is unknown. Given two points of a line one can solve for the slope. Then one can solve for the y-intercept. With the slope and y-intercept one can describe a line in slope-intercept form.

Solved Problem 1.27 For points (0, -2) and (4, 2), find the slope-intercept form of the line

For slope intercept form $y = \boxed{\text{m}}\, x + \boxed{\text{c}}$, first it is necessary to find the slope m and the y-intercept c.

One can choose either of the two points provided to be point 1. Point 1 is $(x_1, y_1) = (0, -2)$ and point 2 is $(x_2, y_2) = (4, 2)$. Recall that $2 - (-2) = 2 + 2 = 4$

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-(-2)}{4-0} = \frac{2+2}{4-0} = \frac{4}{4} = 1 \qquad \boxed{\text{the slope } m \text{ is } 1}$$

Notice that the y-intercept c is now the only unknown. One of the points provided (0, -2) has zero as the x-coordinate, and is the y-intercept. So the y-intercept is -2. Recall that $1 \times x = 1x = x$ and $x + (-2) = x - 2$ $\boxed{\text{Final Answer} \quad y = x - 2}$ ■

Solved Problem 1.28 For points (-4, 3) and (4, 1), find the slope-intercept form of the line

---

For slope intercept form $y = \boxed{\text{m}}\ x + \boxed{\text{c}}$, first it is necessary to find the slope m and the y-intercept c.

Point 1 is $(x_1, y_1) = (-4, 3)$ and point 2 is $(x_2, y_2) = (4, 1)$. Recall that $4 - (-4) = 4 + 4 = 8$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1-3}{4-(-4)} = \frac{-2}{4+4} = \frac{-2}{8} = -\frac{1}{4} \qquad \boxed{\text{the slope is } -\frac{1}{4}}$$

In this case, the y-intercept is not provided. Since at least one point is provided, either point can be used to solve for the y-intercept. The point (-4, 3) will be used, and it is simply plugged into the slope-intercept equation. $x = -4$ and $y = 3$

Recall that $-1 \times -4 = 4$ and $-\frac{1}{4} \times -4 = \frac{1}{4} \times 4 = 1$

$$\boxed{\text{y}} = -\frac{1}{4}\boxed{\text{x}} + c \qquad \boxed{3} = -\frac{1}{4}\boxed{\text{(-4)}} + c \qquad \text{so } 3 = 1 + c$$

Now subtract 1 from both sides.

$$3\ \boxed{\text{-1}} = 1\ \boxed{\text{-1}} + c \qquad \boxed{2 = \text{c}} \qquad \boxed{\text{the y-intercept is } 2}$$

The solution can now be assembled $\boxed{\text{Final Answer} \quad y = -\frac{1}{4}x + 2}$ ■

Slope-intercept form recap, $y = mx + c$:

- m is the slope
- c is the y-intercept
- All that is necessary is the slope and y-intercept
- If the slope is unknown then 2 points on the line are needed to find the slope

Linear functions have been described in point-slope form and slope-intercept form. There is another form called standard form that is not difficult, and it is a slight variation of the previous forms.

Definition 1.9 — Standard Form. Standard form separates terms with variables such as x and y to one side of the equal sign. The constant term c appears on the other side of the equal sign. In standard form, a line is described by the following equation.

$$\boxed{ax+by=c}$$

or

$$\boxed{c=ax+by}$$

- The variables are x and y
- The coefficients of the variables are a and b
- The $\boxed{\text{coefficients should be integers}}$ not fractions

Keep in mind that standard form is now only a slight change from what has already been demonstrated. For example, the final answer from the previous solved problem is shown below. Notice, that the term with the variable x, is shown below, with an orange border. It needs to moved to the left side of the equal sign, to create standard form. Then the fraction must be converted to an integer.

Slope-Intercept form

$$y = \boxed{-\frac{1}{4}x} + 2$$

$$\xrightarrow{\text{convert}}$$

Standard form

$$x + 4y = 8 \quad \text{How?}$$

To achieve this form, we would do the following:

- Move the term $-\dfrac{1}{4}x$ from one side of the equal sign to the other
- "Clear" or convert the fraction $\dfrac{1}{4}$ into an integer

Tip Recall that rational numbers are ratios such as $\frac{1}{8}, \frac{1}{4}, \frac{1}{3}$, ...

Tip Recall that integers are numbers such as -3, 2, 1, 0, 1, 2, 3, ...

In order to move the term $-(1/4)x$ to the left side of the equal sign, the term shown below with an orange border is added to both sides of the equal sign. Why? To move the term $-(1/4)x$ to the other side of the equal sign, it is necessary to do the opposite of what is there, to both sides of the equation.

In other words, notice that the first term that is in an orange border just below has a negative symbol. The opposite of this would be to add it. So it is necessary to add the same exact term to both sides (meaning both sides of the equal sign).

$$y = \boxed{-\frac{1}{4}x} + 2 \qquad y + \boxed{\frac{1}{4}x} = -\frac{1}{4}x + 2 + \boxed{\frac{1}{4}x} \qquad -\frac{1}{4}x + \frac{1}{4}x = 0 \qquad \text{So} \quad y + \frac{1}{4}x = 2$$

A last step is to clear the fraction 1/4. This is because standard form should have integer coefficients, not rational coefficients. It will be necessary to multiply both sides of this equation by 4. Recall that $1 = \frac{4}{4}$

$\frac{1}{4}x + y = 2$ $\qquad$ $\frac{1}{\boxed{4}}x + y = 2$ $\qquad$ The fraction has the denominator 4.

$4\left(\frac{1}{4}x + 1y\right) = 4(2)$ $\qquad$ $\left(\frac{4}{4}x + 4y\right) = 4(2)$ $\qquad$ $(x + 4y) = 8$

The two equations with an orange border just below are in standard form. In the two equations below, the constant is 8. This constant 8 can be on the left side or the right side of the equal sign. Notice that the terms with the variables x and y are on one side, and the constant must be on the other side of the equal sign.

Standard form $\boxed{x + 4y = 8}$ is the same as $\boxed{8 = x + 4y}$

Solved Problem 1.29 Find the least common multiple of the following numbers

(a) 2 and 5 $\qquad$ (b) 4, 6, and 8

(a) For 2 the multiples are 2, 4, 6, 8, $\boxed{10}$, ...

For 5 the multiples are 5, $\boxed{10}$, 15, 20, ... 10 is the lowest number found in both lists.

$\boxed{\text{Final Answer} \quad \text{The lowest common multiple of 2 and 5 is 10}}$

(b) For 4 the multiples are 4, 8, 12, 16, 20, $\boxed{24}$, 28, ...

For 6 the multiples are 6, 12, 18, $\boxed{24}$, 30, 36, ...

For 8 the multiples are 8, 16, $\boxed{24}$, 32, ... 24 is the lowest number found in these lists.

$\boxed{\text{Final Answer} \quad \text{The lowest common multiple of 4, 6, and 8 is 24}}$ ▪

Solved Problem 1.30 For the following equations simply move all terms with variables to one side of the equal sign (x, y, and z are variables)

(a) $y = \frac{3}{4}x + 7$ $\qquad$ (b) $y = \frac{1}{5}x + \frac{3}{7}z + 7$

(a) The term that will be moved is $\boxed{\frac{3}{4}x}$. The opposite of this, $-\frac{3}{4}x$ is applied to both sides. $\boxed{-\frac{3}{4}x}+y=\frac{3}{4}x+7-\boxed{\frac{3}{4}x}$ $\boxed{\text{Final Answer} \quad -\frac{3}{4}x+y=7}$

(b) The FIRST term that will be moved is $\boxed{\frac{1}{5}x}$. The opposite of this, $-\frac{1}{5}x$ is applied to both sides. $\boxed{-\frac{1}{5}x}+y=\frac{1}{5}x+\boxed{-\frac{1}{5}x}+\frac{3}{7}z+7$ $-\frac{1}{5}x+y=\boxed{\frac{3}{7}z}+7$

The SECOND term that will be moved is $\boxed{\frac{3}{7}z}$. The opposite of this, $-\frac{3}{7}z$ is applied to both sides. $-\frac{1}{5}x+y\boxed{-\frac{3}{7}z}=\frac{3}{7}z\boxed{-\frac{3}{7}z}+7$ $\boxed{\text{Final Answer} \quad -\frac{1}{5}x+y-\frac{3}{7}z=7}$ ■

The final answer in the last solved problem has three terms on the left side of the equal sign. These three terms do not have to be in order, but it is common, or a custom, or a convention to put terms in order by the variables, meaning x, y, and z. The last two solved problems will only take a student about 10 minutes to complete, but they show how to work with fractions, least common multiples, and algebraic manipulations. A few minutes with the last two solved problems is time well spent.

Solved Problem 1.31 Rewrite the equation $y=\frac{2}{3}x+7$ in standard form

The term that needs to be moved is $\boxed{\frac{2}{3}x}$. The opposite of this is $\boxed{-\frac{2}{3}x}$, so this is applied to both sides. $\boxed{-\frac{2}{3}x}+y=\frac{2}{3}x\boxed{-\frac{2}{3}x}+7$ $-\frac{2}{3}x+y=7$

Notice, just above that $\frac{2}{3}$ is subtracted from both sides of the equation.

$$\overset{1}{}\overset{2}{} 3\left(\frac{-2}{3}x+1y\right)=3(7) \qquad \frac{-6}{3}x+3y=3(7) \qquad \boxed{\text{Final Answer} \quad -2x+3y=21}$$

The fraction has the denominator 3. So both sides of the equation are multiplied by 3. ■

Standard form recap, $ax+by=c$:

- a and b are integer coefficients

- The coefficients a and b, should not be fractions
- x and y are variables
- Terms with variables are on one side of the equation
- The constant is on the other side of the equation

Table 1.1: Straight Line Forms

Form	General Equation
Point-Slope Form	$y - y_1 = m(x - x_1)$
Slope-Intercept Form	$y = mx + c$
Standard Form	$ax + by = c$

Solved Problem 1.32 Rewrite the equation $ax + by = c$ in slope-intercept form.

Recall slope-intercept form, $y = mx + c$. Slope-intercept form isolates the variable on one side of the equation. In order to do this the ax term can be moved to the other side of the equation. If one subtracts ax from both sides of the equation, it is eliminated from the left side. This is shown below.

$$ax \boxed{-ax} + by = \boxed{-ax} + c \quad \text{leads to} \quad by = -ax + c$$

In by shown, just above, there is multiplication. The variable y needs to be further isolated.

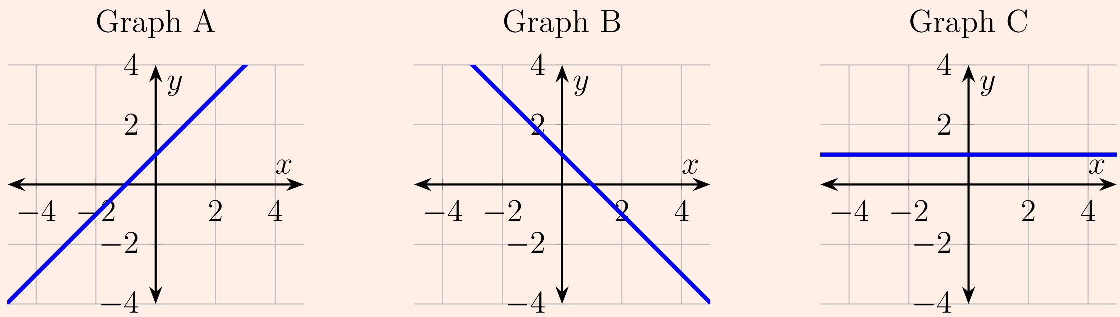
$$\frac{\cancel{b}y}{\cancel{b}} = -\frac{ax}{b} + \frac{c}{b} \qquad \boxed{y = -\frac{a}{b}x + \frac{c}{b}}$$

Recall that multiplication and division are strongly associated. Division is applied to both sides of the equation to eliminate the multiplication in by. This leads to the final answer where the variable y is isolated, and the equation is in slope-intercept form. ■

Solving for an unknown was demonstrated in Section 1.3, but it is worthwhile to review and reinforce what happened in the last solved problem. First, it became necessary to isolate the y variable.

Solved Problem 1.33 The standard form of a line $ax + by = c$ can be rewritten in slope-intercept form as $y = -\frac{a}{b}x + \frac{c}{b}$. Match the following possible conditions for a and b, with the 3 following graphs.

(a) a > 0, b > 0 (b) a < 0, b > 0 (c) a = 0, b > 0



In the slope-intercept equation $y = \boxed{-\frac{a}{b}x} + \frac{c}{b}$, the slope term is shown with an orange border.

(a) a > 0, b > 0 In this case, if both a and b are positive, the slope term is negative. If a and b are positive then the slope term stays negative. Only Graph B has a negative slope. Recall that $negative \times positive = negative$

(b) a < 0, b > 0 In this case, a is negative and b is positive. The slope term has a negative, but a is also negative. This makes the slope term positive. Only Graph A has a positive slope. Recall that $negative \times negative = positive$

(c) a = 0, b > 0 In this case a is zero, and b is positive. Again, the slope term has a negative, but if a is zero, then the slope must be zero. Recall that a denominator cannot be zero, but a numerator can be zero. Only Graph C demonstrates a zero slope.

Notice that the specific values of b and c are never provided. This is unnecessary information. It is possible to match the conditions listed above with the graphs, based on whether the slope is positive, negative, or zero. ■

1.5 Systems of Equations

? What is meant by solving a system of equations?

A system of equations is a group of 2 or more equations. What are the possible outcomes if two lines are described? Two lines can either intersect once, not at all, or the two lines might overlap completely. If the two lines intersect once, then there is one solution. If the two lines do not intersect at all then there is no solution. If the two lines overlap completely then there is an endless list of solutions, and there is said to be infinite solutions.

The purpose with a system of equations is to understand how 2 or more lines are related. Do they intersect? Where do they intersect? The three possible cases for the solution of a system of equations are:

- One solution
- No solution
- Infinite solutions

This is shown, below, in a visual way.

Fig. 1.28: One Solution

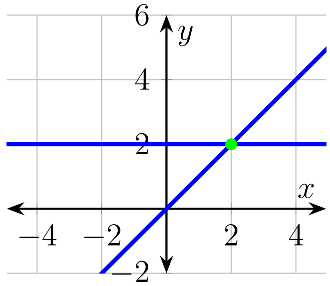


Fig. 1.29: No Solutions

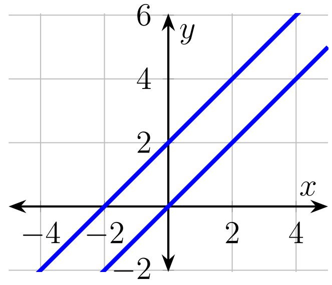
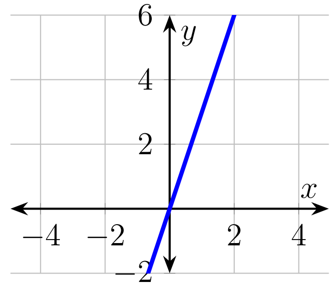


Fig. 1.30: Overlap



The graph in Figure 1.28, above, clearly shows the case where the 2 lines intersect, at one point. There is said to be one solution at the point of intersection. The graph in Figure 1.29 shows the possible case where the two lines never intersect. If two lines never intersect, there is no point of intersection. There is said to be no solution. The graph in Figure 1.30 shows the rare but possible case where two lines overlap completely. In Figure 1.30 there are actually two lines, but if they overlap completely then they would appear as one line. This leads to infinitely many points of intersection. There is said to be infinitely many solutions.

Tip The main question with a system of equations is, "Do the lines intersect?"

A system of equations can be solved by three different methods:

- Graphing
- Substitution
- Elimination

First, one should be familiar with the general layout for a system of equations. Below, on the left, the general form is shown. One can see, below, that a system of equations is simply two or more equations. Below, two examples of systems of linear equations are shown.

A System of Equations	Example 1	Example 2
1: $a_1x+b_1y=c_1$	$1x+0y=0$	$3x+2y=16$
2: $a_2x+b_2y=c_2$	$-3x+1y=2$	$7x+1y=19$

Notice, above that all the equations have the x and y variables to the left of the equal sign. All the equations have the constants to the right of the equal sign. Under the heading "A System of Equations", just above, to the left, a_1 and a_2 represent coefficients. Then, just above, to the left, c_1 and c_2 represent constants. One can see in Example 1, above, that the coefficients are 1, 0, -3, and 1. In Example 1, above, the constants are 0 and 2. In Example 1, above, The equation $1x+0y=0$ can be simplified as $x=0$, and this is used in the example below. This is because $1x=x$ and $0y=0$.

Finding the x-intercept is literally an example of solving a system of equations. A system of equations is shown below. Recall that a solution is simply the point where the two lines intersect. This graph, just below, shows the graphing, visual method of solving linear equations

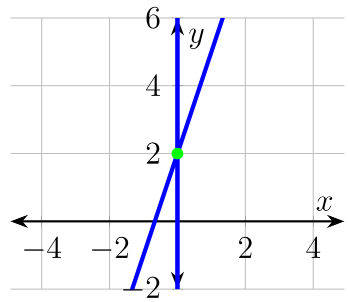
The equation $y = 3x + 2$ can be rewritten in standard form as $-3x + y = 2$. The equation $x = 0$ is already in standard form. Recall that standard form has the variables x and y on one side of the equal sign and the constant on the other side of the equal sign.

A System of Equations

Equation 1: $x = 0$

Equation 2: $-3x + y = 2$

The solution is the point (0, 2) shown in green



Just above the **graphing** method shows that the solution of a system of equations is simply the point of intersection, shown as a green dot.

The **substitution** method works by simplifying a system of equations. Notice that the two equations, shown just below, have something in common. Both equations have the x variable. This can be used to turn two equations into one simple equation.

$$x = 0 \qquad -3x + y = 2$$

To use the substitution method, one notices that the variable x is found in both equations. It is known that $x = 0$. Zero can be plugged into the second equation to find the solution. This is shown below. Recall that $3 \times 0 = 0$

$$-3\boxed{\text{x}} + y = 2 \qquad -3\boxed{0} + y = 2 \qquad 2 = y \qquad \text{So } y = 2 \text{ the solution is } \boxed{(0,\ 2)}$$

The system of equations has just been solved by the substitution method. A point was found that is common to two lines. The meaning of this solution is that the y-intercept is 2. It has been confirmed that these two lines intersect once. What if these lines touch an infinite number of times. The determinant can be used to confirm that the there is only one solution.

Elimination is a third approach to solve systems of equations. In Step 1 below, equation 1 has a $-3x$ and equation 2 has an x. If both sides of equation 2 are multiplied by 3, it becomes $3x = 0$, which is shown in Step 2, below. This will make it possible to eliminate the terms with x.

In Step 3, below the second equation is simply added to the first equation, term by term $-3x$ plus $3x$ equals zero, and y plus zero equals y. Also, $2 + 0 = 2$. This leaves $y = 2$. The solution or point of intersection is then (0, 2).

Step 1	Step 2	Step 3
$-3x+1y=2$ $x \quad =0$	$-3x+1y=2$ $3x \quad =0$	$-3x+1y=2$ $3x \quad =0$ $\boxed{\text{y=2}}$

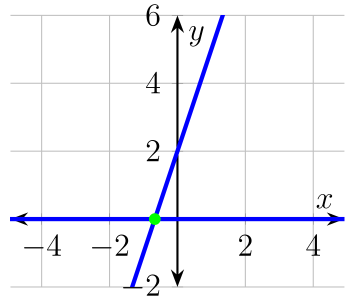
Finding the x-intercept is also a simple example of solving a system of equations. The graphing, substitution, and elimination methods are again demonstrated below.

A System of Equations

Equation 1: $y=0$

Equation 2 $-3x+y=2$

The solution is the point $\left(-\frac{2}{3},0\right)$ shown in green



Just above, the **graphing** method shows that the solution of a system of equations is simply the point of intersection. This point of intersection is the x-intercept, and it is shown as a green dot.

Again, the **substitution** method works by simplifying a system of equations. Notice that the two equations shown just above have something in common. Both equations have the y variable. This can be used to turn two equations into one simple equation.

One notices that the variable y is found in both equations. It is known that $y=0$. This can be plugged into the second equation to find the solution. This is shown below.

$$-3x+\boxed{\text{y}}=2 \qquad -3x+\boxed{0}=2 \qquad -3x=2$$

Now both sides are divided by -3 to solve for x. Recall that $\frac{-3}{-3}=1$

$$\frac{-3x}{\boxed{-3}}=\frac{2}{\boxed{-3}} \qquad x=\frac{2}{-3} \qquad \text{So, } x=-\frac{2}{3} \text{ the solution is then } \left(-\frac{2}{3},0\right)$$

The system of equations has just been solved by the substitution method. The meaning of this solution is that the x-intercept is $-\frac{2}{3}$.

Elimination is a third approach to solve systems of equations. In Step 1 below, both equations have y. So equation 2 is multiplied by -1, so that it becomes $-y=0$, and this is subtracted from the first equation, term by term. One can see in Step 2, that this subtraction leaves $-3x=2$. In Step 3, both sides of $-3x=2$ are divided by 3 to leave $x=-\frac{2}{3}$.

Step 1

$$\begin{aligned} -3x + y &= 2 \\ y &= 0 \end{aligned}$$

Step 2

$$\begin{array}{rcl} -3x + y &=& 2 \\ -y &=& 0 \\ \hline -3x \quad &=& 2 \end{array}$$

Step 3

$$\frac{-3x}{3} = \frac{2}{3}$$

$$\boxed{x = -\frac{2}{3}}$$

Solved Problem 1.34 Solve the system of linear equations with the following equations $2x + y = 15$ and $3x - y = 5$, by substitution.

It is important to see the two equations in standard form as a system of equations. Substitution makes it possible to turn two equations into one equation.

$$2x + y = 15 \qquad\qquad 3x - y = 5$$

The second equation above will be written in terms of y. To do this the other terms will be moved to the right of the equal sign. To move the term $3x$, the opposite is added to both sides.

$$3x - y = 5 \qquad 3x \boxed{-3x} - y = 5 \boxed{-3x} \qquad \boxed{y = 5 - 3x} \quad \text{or} \quad y = -3x + 5$$

This second equation in terms of y can be plugged into y of the first equation. Then the equation is simplified.

$$2x + \boxed{\text{y}} = 15 \qquad 2x + \boxed{(5 - 3x)} = 15 \qquad 2x + 5 - 3x = 15 \qquad 2x - 3x + 5 = 15$$

This leaves an equation with one variable, shown below. Also, 5 is moved to the right side of the equal sign by subtracting it from both sides

$$-1x + 5 = 15 \qquad 5x + 5 \boxed{-5} = 15 \boxed{-5} \qquad -x = 10 \qquad \boxed{x = -10}$$

Recall that the trick was to turn two equations into one equation that only had the variable x. It is still necessary to solve for y. The value for x is just plugged into an equation as shown below.

$$3 \boxed{x} - y = 5 \qquad 3(\boxed{-10}) - y = 5 \qquad -30 - y = 5$$

Notice below, that y is added to both sides in order to move the y to the right side of the equal sign. Also, 5 is moved to the left side of the equal sign by subtracting it from both sides.

$$-30 - y + y = 5 + y \qquad \boxed{-30 = 5 + y} \qquad -30 - 5 = 5 - 5 + y \qquad \boxed{-35 = y}$$

$$\boxed{\text{Final Answer} \qquad (-10, -35)}$$

■

When solving a system of linear equations it is possible to check your solution. For example for the solved problem above, the solution is the point (-10, 35). This is simply plugged into the system of equations.

$$2x+y=15 \qquad\qquad 3x-y=5$$

First plug (-10, 35) into $2\boxed{\text{x}}+\boxed{\text{y}}=15$ $\quad 2(-10)+(-35)=15$

$-20+(-35)=15$ which is correct!

Next plug (-10, 35) into $3\boxed{\text{x}}-\boxed{\text{y}}=5$ $\quad 3(-10)+(35)=5$

$-30+(35)=5$ which is correct! So the solution (-10, 35) is correct!

Recall, that the determinant can be used to confirm that there is only one solution.

Solved Problem 1.35 Solve the system of linear equations with the following equations $3x+2y=16$ and $7x+y=19$, by substitution.

Substitution makes it possible to turn two equations into one equation.

$$3x+2y=16 \qquad\qquad 7x+y=19$$

The second equation above will be written in terms of y. To do this the other terms will be moved to the right of the equal sign. To move the term $7x$, the opposite is added to both sides.

$7x+y=19 \qquad 7x\boxed{-7x}+y=19\boxed{-7x} \qquad \boxed{y=19-7x}$ or $y=-7x+19$

This second equation in terms of y can be plugged into y of the first equation. Then distribution is applied.

$3x+2\boxed{y}=16 \qquad 3x+\overset{1\ \ 2}{2(19-7x)}=16 \qquad 3x+38-14x=16$

Distribution leaves an equation with one variable, shown below. Also, 38 is moved to the right side of the equal sign by subtracting it from both sides

$3x+\boxed{38-14x}=16 \qquad 3x-14x+38=16 \qquad -11x+38\boxed{-38}=16\boxed{-38}$

$-11x=-22$

Then both sides are divided by -11 to solve for x

$$\frac{-11x}{\boxed{-11}}=\frac{-22}{\boxed{-11}} \qquad x=\frac{-22}{-11} \qquad \boxed{x=2}$$

Recall that the trick was to turn two equations into one equation that only had the variable x. It is still necessary to solve for y. This can be done with either of the equations provided, initially. The value for x is just plugged in as shown below.

$$7\boxed{(x)}+y=19 \qquad 7\boxed{(2)}+y=19 \qquad 14+y=19$$

Notice below, that 14 is subtracted from both sides in order to move the 14 to the right side of the equal sign.

$$14-14+y=19-14 \qquad \boxed{y=5} \qquad \boxed{\text{Final Answer} \quad (2,5)}$$

■

Solved Problem 1.36 Solve the system of linear equations with the following equations $3x+2y=6$ and $5x-2y-10$, by elimination.

Elimination also makes it possible to turn two equations into one equation. The first equation has the term $2y$, and the second equation has the term $-2y$.

$$3x+2y=6 \qquad\qquad 5x-2y=10$$

Simply adding the two equations will eliminate the terms with the variable y

Step 1

$$\begin{array}{l} 3x\ \boxed{+2\text{y}} = 6 \\ 5x\ \boxed{-2y} = 10 \\ \hline \boxed{8\text{x} \qquad = 16} \end{array}$$

Step 2

$$\frac{\cancel{8}\ x}{\boxed{\cancel{8}}} = \frac{\cancel{8}\ (2)}{\boxed{\cancel{8}}}$$

$$\boxed{x=2}$$

It is still necessary to solve for y. This can be done with either of the equations provided, initially. The value for x is just plugged in as shown below.

$$3x+2y=6 \qquad 3\boxed{(x)}+2y=6 \qquad 3\boxed{(2)}+2y=6 \qquad 6+2y=6$$

Next 6 is subtracted from both sides in order to move it to the right side of the equal sign. Then both sides are divided by 2 to solve for y.

$$6-\boxed{6}+2y=6-\boxed{6} \qquad \frac{\cancel{2}\ y}{\boxed{\cancel{2}}} = \frac{0}{\boxed{2}} \qquad \boxed{y=0} \qquad \boxed{\text{Final Answer} \quad (2,\ 0)}$$

■

Solved Problem 1.37 Solve the system of linear equations with the following equations $x+y=-7$ and $4x+2y=-18$, by elimination.

Elimination makes it possible to turn two equations into one equation. One can choose to eliminate the x term or the y term.

$$4x+2y=-18 \qquad\qquad x+y=-7$$

If the second equation is multiplied by -2 as shown below, then it can be used to eliminate the y term of the equations.

$$\boxed{\text{-2}}(x+y)=\boxed{-2}(-7) \quad -2(x+y) \qquad -2x-2y=14$$

Step 1

$$\begin{aligned} 4x+2y&=-18\\ -2x-2y&=\ \ 14 \end{aligned}$$

$$\boxed{2x \qquad =-4}$$

Step 2

$$\frac{\cancel{2}\,x}{\cancel{2}}=\frac{\cancel{2}\,(-2)}{\cancel{2}}$$

$$\boxed{x=-2}$$

It is still necessary to solve for y. This can be done with either of the equations provided, initially. The value for x is just plugged in as shown below.

$$\boxed{\text{x}}+y=-7 \qquad \boxed{-2}+y=-7 \qquad -2+2+y=-7+2 \qquad \boxed{y=-5}$$

$$\boxed{\text{Final Answer} \qquad (-2,-5)}$$

1.6 Interpret and Apply

This chapter is about recognizing and applying algebra of linear functions. This is about being able to describe activities, behaviors, and systems. One might describe activity in a business, factory, laboratory, restaurant, or resort. A system could be as simple as a coffee machine, a microwave, or a swimming pool. Algebra is just another language, and with practice one can understand and communicate in this language.

Solved Problem 1.38 A group of students is holding a fundraiser. They are charging \$10 for each car wash. The group of students will wash 4 cars per hour What is their total revenue, after a 5-hour fundraiser?

Key components of this problem would be:

- 4 cars per hour or 4 $\dfrac{cars}{hour}$
- The fundraiser will last 5 hours
- \$10 per car or 10 $\dfrac{dollars}{car}$
- Revenue?

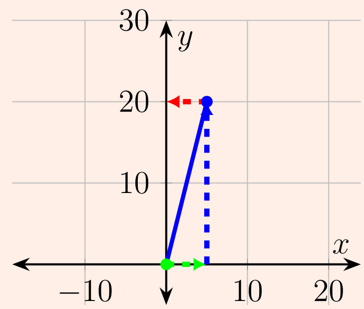
Revenue is just the money or income generated by the sale of goods or services.

The objective is to find revenue, so the final answer will be measured in dollars. If there are 5 hours and every hour 4 cars are washed, and every single car wash brings in \$10 then these quantities are simply multiplied.

$$5\cancel{hours} \times 4\frac{\cancel{cars}}{\cancel{hour}} \times 10\frac{dollars}{\cancel{car}} = \$200 \qquad 5 \times 4 \times \$10 = \$200 \qquad 20 \times \$10 = \$200$$

Consider the units involved. When units are multiplied, as shown above, components of units cancel. Hours cancels hour. Cars cancels car. Only dollars, is not cancelled, so the final units will be dollars $\quad \cancel{hours} \times \frac{\cancel{cars}}{\cancel{hour}} \times \frac{dollars}{\cancel{car}}$

- $4\,\frac{cars}{hour}$ is a linear function
- $4\,\frac{cars}{hour}$ is a rate with a slope
- $slope = \frac{rise}{run} = \frac{20}{5} = 4$
- The graph shows the linear function $y = 4x$



$\boxed{\text{Final Answer} \qquad \text{The total revenue is \$200}}$

Recall that in order to describe a line or a linear function one needs, either:

- two points
- or one point and the slope

A linear function can describe a car wash. So, if a slope and one point is found or recognized, then the linear function is found or recognized. The last solved problem provided the slope of 4 cars per hour. The point may not seem so easy to find, at first. Any event usually has a starting point, and this means that usually the point (0, 0) can be used. This means that at time zero, or at the beginning, zero cars have been washed. So, with a slope of 4 and a point (0, 0) the linear function $y = 4x$ or $f(x) = 4x$ is found.

In the graph just above, the x-axis shows hours. The car wash only lasts 5 hours, so the graph stops at 5 on the x-axis. The y-axis represents total cars washed, and the graph shows with a blue dot that a total of 20 cars were washed.

Cars per hour and dollars per hour describe a velocity or a rate of change. Cars per hour tells us how fast or slow the students wash cars. Dollars per hour tells us how fast or slow the fundraiser is generating revenue. A velocity or rate of change means there is a slope.

The linear function $y = 4x$ can also be written $f(x) = 4x$. The notation $f(x)$ is another way of writing y, which is the output. So, $f(x)$ is basically y. This last notation is read f of x equals 4x, and this is saying that the function of x is 4x. This can sound complicated, but recall that a linear function can be thought of as a simple system with one input and one output. In the graph above, the blue dashed line also illustrates input, and the red dashed line illustrates output. Note that if the input is 5 the output is 20.

Tip Cars per hour, or dollars per hour, or miles per hour, or kilometers per hour are rates of change. A rate of change is a slope.

The last solved problem shows how a linear function can be used to describe a car wash. A car wash is a simple activity and the linear function $y = 4x$ or $f(x) = 4x$, is also a simple function.

The last solved problem considered a simplified version of a car wash. One can consider and argue that a car wash may have slower days. One can consider and argue that a car wash will likely have busier days, on the weekends. The car wash might have busier months. The constant rate of change used in the last solved problem, is ok for a simplified version of a car wash.

Solved Problem 1.39 The revenue of a restaurant is described by the linear function $f(x) = 500x$, where x is days. What will be the total revenue after 30 days?

$f(x) = 500x$ is the same as $y = 500x$. This linear function can be treated as a simple system where the input is days and the output is total revenue. Days are simply plugged into x, which is shown with an orange border below.

$$y = 500\ \boxed{(x)} \qquad y = 500 \times \boxed{30} \qquad y = 1500$$

$$\xrightarrow{\text{Input } x = 30 \text{ days}} \boxed{\quad y = 500x \quad} \xrightarrow{\text{Output } y = \$1500}$$

If the input is 30 days, then the output is 500×30, which is equal to 1500 or 1500 dollars. One can also review units as shown below. Notice on the right side of the equal sign that days cancels day, and only the units dollars are left.

$$y\ dollars = 500\ \frac{dollars}{day}\ x\ days \qquad dollars = \frac{dollars}{\cancel{day}}\cancel{days}$$

$\boxed{\text{Final Answer} \qquad \text{The total revenue is \$1500}}$ ■

Solved Problem 1.40 If a swimming pool fills at 600 gallons per hour, describe this with a linear function.

Gallons per hour is a rate, so it is also a slope. To arrive at a linear function one needs either two points or a slope and one point. Since the activity of filling a swimming pool has a starting point. This starting point can be assigned the point (0, 0).

If the point (0, 0) can be used then the y-intercept is 0, or the same (0, 0) point. This means that at zero hours there are zero gallons of water in the pool. In other words the pool simply starts out as empty. So at this point the slope and y-intercept can be used to assemble the linear function, shown below. $y = 600x$ or $f(x) = 600x$

$$\xrightarrow{\text{Input } x} \boxed{f(x) = 600x} \xrightarrow{\text{Output } y}$$

For this linear function the input would be hours, and the output would be total gallons.

$\boxed{\text{Final Answer} \quad \text{The linear function is} \quad f(x) = 600x}$ ■

Solved Problem 1.41 A swimming pool fills at 600 gallons per hour, but it already has 600 gallons of water in it. Describe this with a linear function.

This is mostly the same as the previous solved problem. The difference is that the pool begins with water already in it. To begin with the pool has 1000 gallons of water.

To say that there is initially 1000 gallons of water, and then more water will be added is to say $y = 600 + 600x$ or $y = 600x + 600$, as shown below.

$$\xrightarrow{\text{Input } x} \boxed{f(x) = 600x + 600} \xrightarrow{\text{Output } y}$$

Again for this linear function the input would be hours, and the output would be total gallons.

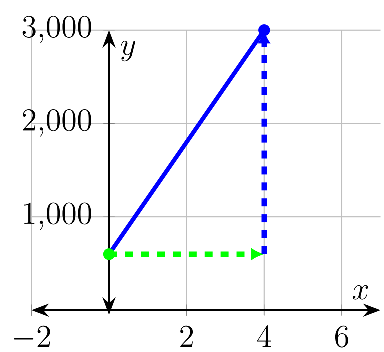
$\boxed{\text{Final Answer} \quad \text{The linear function is} \quad f(x) = 600x + 600}$ ■

$f(x) = 600x + 600$ is shown to the right

The y-intercept is shown as a green dot

$$slope = \frac{rise}{run} = \frac{2400}{4} = 600$$

At 4 hours the pool has 3 thousand gallons. This is shown with a blue dot.



In this last solved problem, one can see real, applied meaning for a y-intercept. In the last solved problem, the function is $f(x) = 600x + 600$, and this function is in slope intercept form. The y-intercept is (0, 600). In other words, at the beginning there is already 600 gallons of water. It takes practice to apply and interpret linear functions.

Solved Problem 1.42 A swimming pool drains at 2000 gallons per hour. Describe this with a linear function.

Again, there is a rate of change, but it involves a decrease. The slope would be $-2,000$. It is negative because this describes a decrease. The problem does not mention how much water is in the pool. So, one can only describe how the pool drains. The starting point can be assigned the point (0, 0). The linear function would be $f(x) = -2000x$.

$$\xrightarrow{\text{Input } x} \boxed{f(x) = -2000x} \xrightarrow{\text{Output } y}$$

Again, for this linear function the input would be hours, and the output would be total gallons.

$\boxed{\text{Final Answer} \quad \text{The linear function is} \quad f(x) = -2000x}$ ■

Solved Problem 1.43 An individual should be administered a particular medicine at a dose of $\frac{1}{2}$ a milligram per kilogram of body weight. Describe this with a linear function.

The word "per" again describes a rate of change or a slope. The slope here is positive because as the weight of the individual increases, the necessary dosage increases. The slope here is simply $\frac{1}{2}$. The linear function is $f(x) = \frac{1}{2}x$

$$\xrightarrow{\text{Input } x} \boxed{f(x) = \frac{1}{2}x} \xrightarrow{\text{Output } y}$$

For this linear function, the input would be kilograms, and the output would be total milligrams or mg. Recall that milligrams are abbreviated as mg, and kilograms are abbreviated as kg.

$\boxed{\text{Final Answer} \quad \text{The linear function is} \quad f(x) = \frac{1}{2}x}$ ■

In this last solved problem, the function, like a tool, makes it possible to know how much medicine to administer. One enters the individuals weight into this function, and like a handy tool it outputs the needed dosage.

Solved Problem 1.44 An event of several days will require 40 hamburgers per day. Describe this with a linear function.

Again the word "per" describes a rate of change or a slope. The slope for hot dogs is positive because the number of hot dogs needed increases every day. The linear function could be written in the following way, and it would be 100% correct.

$$\text{total} = 40\,\frac{\text{hamburgers}}{\text{day}} \times \text{number of days}$$

It is clear that the rate of change or slope is 40 hamburgers per day. Also, the variable x can be assigned to the number of days. Then the linear function is written as $f(x) = 40x$

$$\text{Input } x \longrightarrow \boxed{f(x) = 40x} \quad \text{Output } y \longrightarrow$$

The output would be the total number of hamburgers need for the entire event.

$\boxed{\text{Final Answer} \quad \text{The linear function is} \quad f(x) = 40x}$ ■

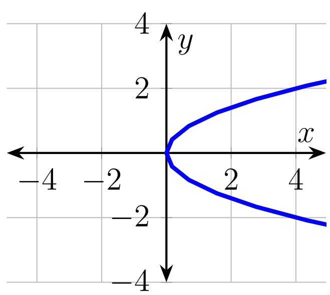
The linear function in this last solved problem could be used like a tool, by event organizers. With the function, in this last solved problem, event organizers can make sure that they have enough hamburgers for every day of the event. The purpose of a function can be to provide an output of interest, when it is given the appropriate input.

1.7 Linear Functions Review Problems

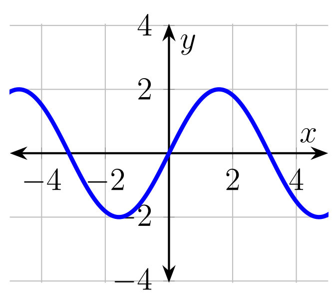
Click on the page number to the right to see the chapter discussion. Click on the solution number to navigate to the solution.

1. Which of the graphs below is a function? (Page 19) (Solution 1)

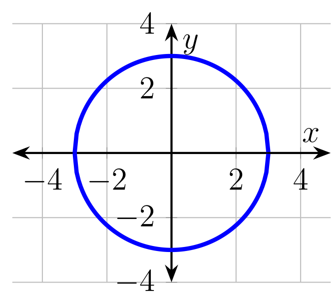
Graph A



Graph B

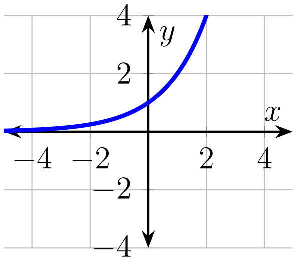


Graph C

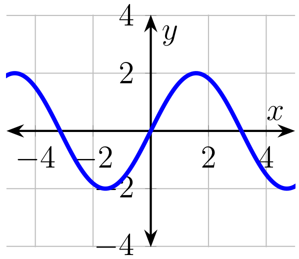


2. Which of the graphs below is linear? (Page 18) (Solution 2)

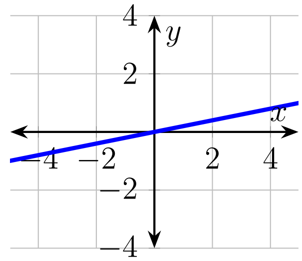
Graph D



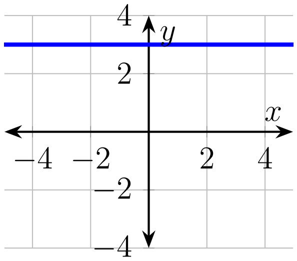
Graph E



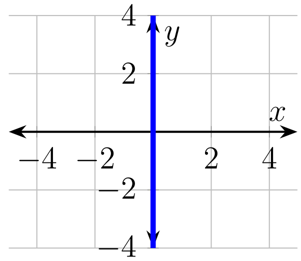
Graph F



3. What is the slope of the graph below? (Page 21) (Solution 3)



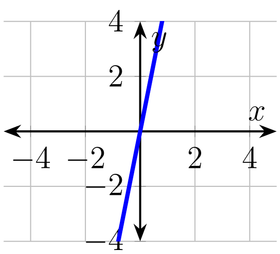
4. What is the slope of the graph below? (Page 33) (Solution 4)



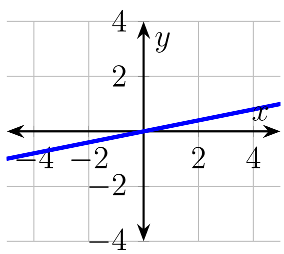
5. Match the graphs below with the following equations.

$f(x) = x$ $\qquad f(x) = \frac{1}{5}x$ $\qquad f(x) = 5x$ (Page 39) (Solution 5)

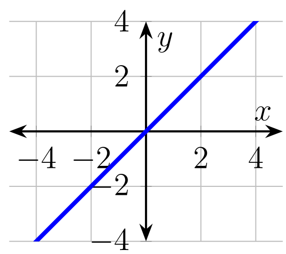
Graph A



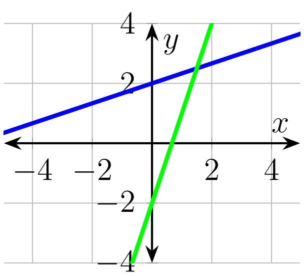
Graph B



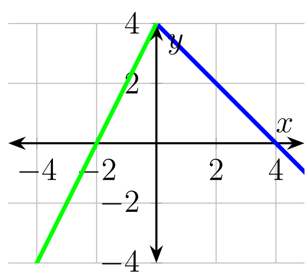
Graph C



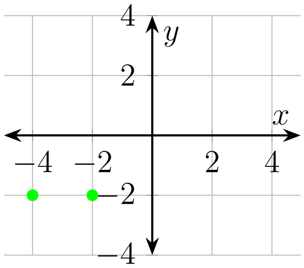
6. Find the y-intercepts for the lines in the graph. (Page 22) (Solution 6)



7. Find the x-intercepts for the lines in the graph. (Page 22) (Solution 7)



8. Which quadrant contains the green points? (Page 16) (Solution 8)



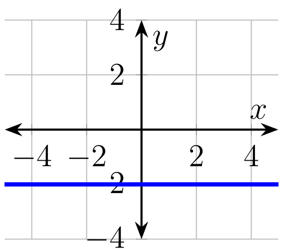
9. Name 3 ways or forms that can be used to describe a linear function.

   (Page 38)
   (Page 34)
   (Page 42)
   (Solution 9)

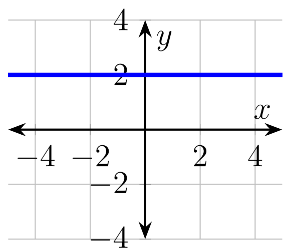
10. Match the graphs below with the following equations:

    $f(x) = -3$ $f(x) = -2$ $f(x) = 2$ (Page 22) (Solution 10)

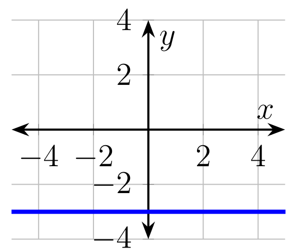
Graph A



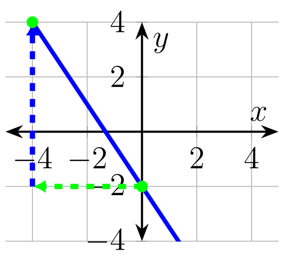
Graph B



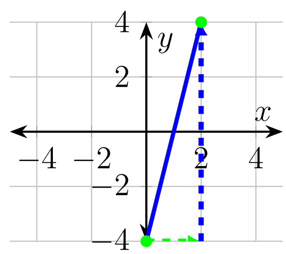
Graph C



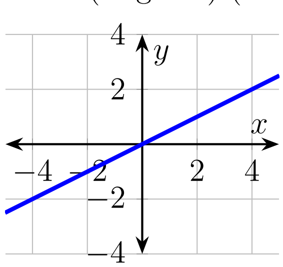
11. Find the slope of the linear function in the graph below. (Page 31) (Solution 11)



12. Find the slope of the linear function in the graph below. (Page 31) (Solution 12)

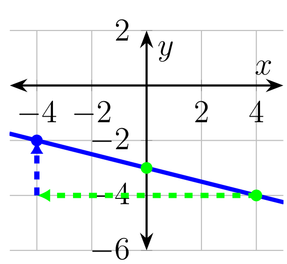


13. Given the function below find the outputs that correspond to the following inputs:
    (a) input = -4 (b) input = 4
    (Page 25) (Solution 13)

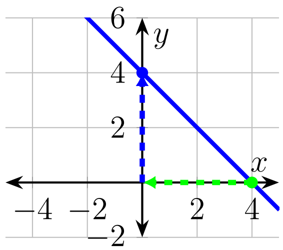


14. Find the point-slope form of the linear function in the graph below.

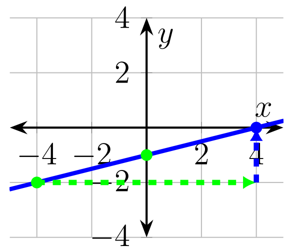
    (Page 34) (Solution 14)



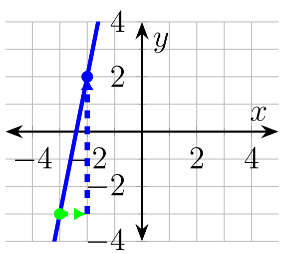
15. Find the slope-intercept form of the linear function in the graph below.
(Page 34) (Solution 15)



16. Find the slope-intercept form of the linear function in the graph below.
(Page 38) (Solution 16)



17. Find the slope-intercept form of the linear function in the graph below.
(Page 38) (Solution 17)



18. Which of the following functions is/are linear function(s) ?
(Page 18) (Solution 18)

(a) $y = 13x + 5$
(b) $y = x^3 + 2$
(c) $y = \frac{1}{6}x + 9$
(d) $y = -x^2$

19. Find the point-slope form of the line described by the two following points:
(1, 6) and (2, 1) (Page 34) (Solution 19)

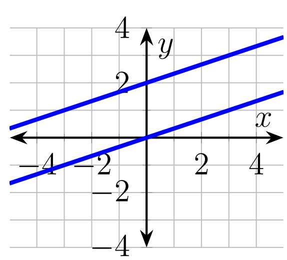
20. Convert $y = \frac{1}{5}x + 12$ into standard form. (Page 44) (Solution 20)

21. Convert $y = -\frac{1}{4}x + 1$ into standard form. (Page 44) (Solution 21)

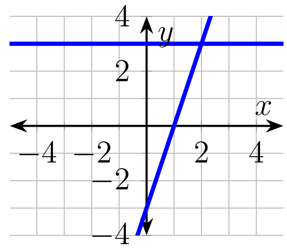
22. A linear function has an input and output nature. For the coordinate pair (5, 12), what is the input, and what is the output? (Page 25) (Solution 22)

23. A linear function has an input and output nature. For the coordinate pair (-4, 16), what is the input, and what is the output? (Page 25) (Solution 23)

24. Find the solution of the system of equations shown in the graph below
(Page 46) (Solution 24)



25. Find the solution of the system of equations shown in the graph below



(Page 46) (Solution 25)

26. Find the solution of the following system of equations. (Page 50) (Solution 26)
$$2x - 3y = -6 \qquad x + 3y = 6$$

27. Find the solution of the following system of equations. (Page 50) (Solution 27)
$$2x - 3y = 0 \qquad x + 3y = 36$$

28. An individual should be administered a particular medicine at a dose of $\frac{1}{4}$ of a milligram per kilogram of body weight, or $\frac{1}{4}\left(\frac{mg}{kg}\right)$ of body weight. (Page 57) (Solution 28)
    (a) Describe this with a linear function
    (b) What is the y-intercept of this linear function?
    (c) What is the rate of change or slope in this word problem?
    (d) What would be the appropriate dose for an individual that weights 80 kg?

29. A swimming pool has 1000 gallons of water in it, to begin with. Thereafter, this pool can be filled at a rate of 700 gallons per hour. (Page 55) (Solution 29)
    (a) Describe this with a linear function
    (b) What is the y-intercept of this linear function?
    (c) What is the rate of change or slope in this word problem?
    (d) How much water will be in the swimming pool if water is added for 3 hours?

30. A coffee shop starts a Monday with 100 kilograms of coffee. Thereafter, 2 kilograms of coffee are being consumed per hour. (Page 55) (Solution 30)
    (a) Describe this with a linear function
    (b) What is the y-intercept of this linear function?
    (c) What is the rate of change or slope in this word problem?
    (d) How much coffee will the coffee shop still have after 4 hours?

1.8 Linear Functions Review Solutions

1. Graph B
2. Graph F
3. The slope is zero
4. The slope is undefined
5. Graph A is $f(x) = 5x$ Graph B is $f(x) = \frac{1}{5}x$ Graph C is $f(x) = x$
6. Blue line: y-intercept is 2 Green line: y-intercept is -2
7. Blue line: x-intercept is 4 Green line: x-intercept is -2
8. Quadrant 3
9. Point-Slope Form, Slope-Intercept Form, Standard Form
10. Graph A is $f(x) = -2$ Graph B is $f(x) = 2$ Graph C is $f(x) = -3$
11. The slope is $-\frac{3}{2}$
12. The slope is 4
13. (a) output = -2 (b) output = 2
14. $y+3 = -\frac{1}{4}(x-0)$ or $y+3 = -\frac{1}{4}x$
15. $f(x) = -x+4$
16. $f(x) = \frac{1}{4} - 1$
17. $f(x) = 5x+12$
18. Only (a) and (c) are linear functions
19. $y-6 = -5(x-1)$
20. $x-5y = -60$
21. $x+4y = 4$
22. Input is 5, Output is 12
23. Input is -4, Output is 16
24. No solution
25. The solution is the point of intersection, (2, 3)
26. The solution is the point of intersection, (0, 2)
27. The solution is the point of intersection, (12, 8)
28. (a) $f(x) = \frac{1}{4}x$ (b) (0, 0) (c) $\frac{1}{4}$ or $\frac{1}{4}\left(\frac{mg}{kg}\right)$ (d) $20mg$
29. (a) $f(x) = 700x+1000$
    (b) (0, 1000) The beginning of an event may be called time = zero, or zero time. At the beginning of this event there are 1000 gallons in the pool, to begin with. For this y-intercept, the x-coordinate is zero, and the y-coordinate is 1000.
    (c) 1000 or $1000\left(\frac{\text{gallons}}{\text{hour}}\right)$
    (d) $f(3) = 2100+1000 = 3{,}100$ gallons so 3,100 gallons

30. (a) $f(x) = -2x + 100$

(b) (0, 100) The beginning of an event may be called time = zero, or zero time. At the beginning of this event there are 100 kilograms of coffee, to begin with. For this y-intercept, the x-coordinate is zero, and the y-coordinate is 100.

(c) -2 or $-2\left(\frac{\text{kilograms}}{\text{hour}}\right)$

(d) $f(x) = -(2 \times 4) + 100 = 92$ gallons so 92 kilograms

Chapter 2: Inequalities and Quadratics

OVERVIEW

The sections of this chapter are:

2.1 Inequalities
2.3 Sets and Intervals
2.2 Graphing Inequalities
2.4 Nonlinear Functions
2.5 Quadratics

Inequality is inherent in our perception - our eyes discern colors, shades, depths, and heights, while our ears detect variations in sound. These differences are communicated through inequalities, necessary for expressing variations in weight, temperature, and value. Quadratics, exemplified by the parabolic trajectory of a kicked or thrown ball, are familiar experiences for most people, mirrored in the arc of objects in motion. Water fountains, with their upward-spraying streams forming inverted parabolas, offer another common encounter with this mathematical concept.

OBJECTIVES

By the end of the chapter, a student will be able to:

- Describe inequalities, sets, and intervals in new ways
- Recognize and describe nonlinear functions
- Solve for an unknown variable
- Multiply binomials
- Apply the sign recognition method to solve quadratic equations
- Apply the AC method to solve quadratic equations
- Apply factoring formulas to factor and solve quadratic equations

2.1 Inequalities

Algebraic inequalities simply describe a difference like a difference in size, weight, length, temperature, or some value. We know that the temperature in a hot kitchen oven is higher than the temperature in a freezer. This can be communicated with an algebraic inequality.

Temperature in a hot kitchen oven $>$ temperature in a freezer

Tip The $>$ symbol means "greater than". It means that the quantity on the left is greater than the quantity on the right.

Tip The $<$ symbol means "less than". It means that the quantity on the left is less than the quantity on the right.

The symbol $>$ has an open side and a closed, pointy side. The open side of the symbol tells which side is greater. The closed, pointy side tells which side is lower. With this understanding the following statements would be true.

weight of a car $>$ weight of a bicycle

temperature of an ice cube $<$ temperature of the sun

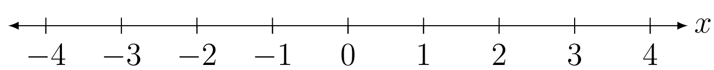
Tip The $\geq$ symbol means "greater than or equal to". It means that the quantity on the left is greater than or equal to the quantity on the right.

Tip The $\leq$ symbol means "less than or equal to". It means that the quantity on the left is less than or equal to the quantity on the right.

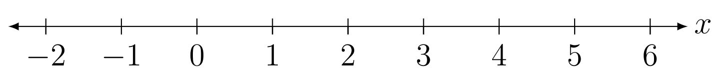
cell phone price $\geq$ \$500 delivery time $\leq$ 24 hours

What cell phone brands and models have a price that is greater than or equal to \$500? Are there cases where the delivery time for a product or service is less than or equal to 24 hours? If one is able to understand these questions one is able to understand the full meaning of the inequalities just above. Inequalities can also be illustrated with a number line.

Definition 2.1 — Number Line. A number line extends infinitely in both positive and negative directions, with increasing numbers to the right and decreasing to the left, but usually, we only focus on a small part of it.



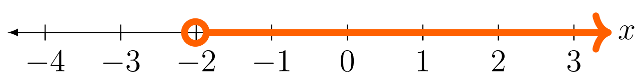
Still, a number line will almost always display a window or sample.



The number line, above, shows numbers between about -2 and 6.

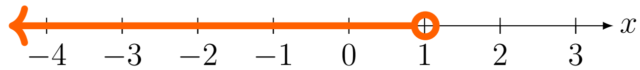
In the definition, just above, both number lines, are labeled with the variable x which is common. However, it could also be labeled with the variable y or z, or any other label.

$2 < x$



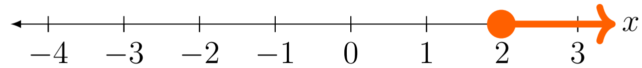
The orange line begins at -2, and it increases in the positive direction. Notice, just above, that a **hollow or open** orange dot occurs at -2. This means that the value -2 is **not** included. Only values of x that are greater than -2, are included. This, is described by inequality $2 < x$.

$x < 1$



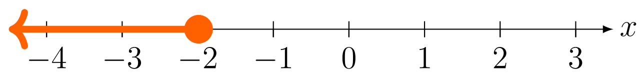
The number line above indicates that x is less than 1, which is $x < 1$ or $1 > x$. The **hollow or open** orange dot occurs at 1. which means 1 is **not** included.

$2 \leq x$



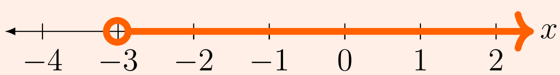
The number line above indicates that x is greater than or equal to 2, which $2 \leq x$ or $x \geq 2$. that a **closed** orange dot occurs at 2 means that 2 is included.

$x \leq 2$

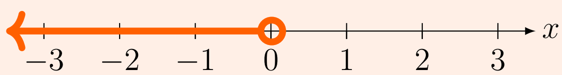


The number line just above illustrates the case where x is less than or equal to -2. This can be written as $x \leq 2$ or $x \leq 2$. Notice just above that a **closed** orange dot occurs at -2. This means that the value -2 is included.

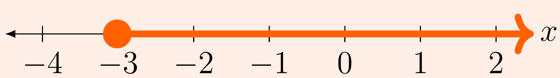
Solved Problem 2.1 Write the inequality for each example below.



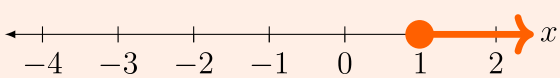
Final Answer $-3 < x$



Final Answer $x < 0$



Final Answer $-3 \leq x$



Final Answer $1 \leq x$

With inequalities, solving for an unknown can lead to adding or subtracting "across" the inequality symbols. Inequality symbols include: $>$, $<$, $\geq$, and $\leq$. The following points need to be repeated and practiced.

Tip In order to solve for an unknown, it is necessary to ISOLATE the unknown if we have to.

Tip Subtraction is another form of addition. Subtraction can be used to eliminate addition, and vise versa.

Tip Division is another form of multiplication. Division can be used to eliminate multiplication, and vise versa.

Tip Whatever change is made to one side of an equation, must also be applied to the other side of the equation.

In the expression, $x - 10 > 2$, the variable x is not alone. In order to isolate the variable x it is necessary to add 10 to both sides.

$$x - 10 > -8 \qquad x - 10 \boxed{+10} > -8 \boxed{+10} \qquad \boxed{x > 2}$$

Just as in linear equations, isolating the variable solves the linear inequality.

Likewise, in the expression, $x + 5 > 8$, the variable x is not alone. In order to isolate the variable x it is necessary to subtract 5 from both sides.

$$x + 5 < 8 \qquad x + 5 \boxed{-5} < 8 \boxed{-5} \qquad \boxed{x < 3}$$

Tip When adding or subtracting across an inequality the inequality stays the same, and is not reversed.

Solved Problem 2.2 Isolate the x variable in the following inequalities

(a) $x + 10 \geq 4$ (b) $x - 12 < 3$ (c) $x - 52 \leq -26$

(a) $x + 10 \geq 4 \qquad x + 10 \boxed{-10} \geq 4 \boxed{-10} \qquad \boxed{\text{Final Answer } x \geq -6}$

(b) $x - 12 < 3 \qquad x - 12 \boxed{+12} < 3 \boxed{+12} \qquad \boxed{\text{Final Answer } x < 15}$

(c) $x - 52 \leq -26 \qquad x - 52 \boxed{+52} \leq -26 \boxed{+52} \qquad \boxed{\text{Final Answer } x \leq 26}$ ■

The steps for reverse-PEMDAS are summarized below.

S - Subtraction	If subtraction exists, then eliminate it.
A - Addition	If addition exists, then eliminate it.
D - Division	If division exists, then eliminate it.
M - Multiplication	If multiplication exists, then eliminate it.
E - Exponents	If an exponent exists, then eliminate it.
P - Parentheses	If parentheses exist, then eliminate them.

In the expression, $(x/2) > 4$, the variable x is not alone. In order to isolate the variable x it is necessary to multiply both sides by 2. Since both sides are multiplied by a positive value, the inequality symbol, $>$, is not reversed.

$$\frac{x}{2} > 4 \qquad \frac{x}{\cancel{2}}\boxed{\times\cancel{2}} > 4\ \boxed{\times 2} \qquad \boxed{x > 8}$$

In the expression, $-5x < -60$, Since both sides are divided by a negative value, the inequality symbol, $<$, is reversed to $>$. This is shown below.

$$\text{Recall that } \frac{\cancel{-5}x}{\cancel{-5}} = x \text{ and } \frac{\cancel{-5}(12)}{\cancel{-5}} = 12 \qquad -5x < -60 \qquad \frac{\cancel{-5}(x)}{\boxed{\cancel{-5}}} > \frac{\cancel{-5}(12)}{\boxed{\cancel{-5}}} \qquad \boxed{x > 12}$$

Tip When multiplying or dividing across an inequality by a negative number the inequality is reversed.

Solved Problem 2.3 Isolate the x variable in the following inequalities

(a) $-2x \geq 100$ (a) $-7x \leq 35$ (a) $\dfrac{x}{-13} < -2$

(a) $-2x \geq 100$ $\quad \dfrac{\cancel{-2}(x)}{\boxed{\cancel{-2}}} \leq \dfrac{\cancel{-2}(50)}{\boxed{\cancel{-2}}} \quad \boxed{\text{Final Answer } x \leq -50}$

(b) $-7x \leq 35$ $\quad \dfrac{\cancel{-7}(x)}{\boxed{\cancel{-7}}} \geq \dfrac{\cancel{-7}(5)}{\boxed{\cancel{-7}}} \quad \boxed{\text{Final Answer } x \geq -5}$

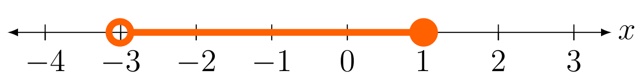
(c) $\dfrac{x}{-13} < -2$ $\quad \left(\dfrac{x}{\cancel{-13}}\right) \times \boxed{\cancel{-13}} > (-2)\boxed{\text{(-13)}} \quad \boxed{\text{Final Answer } x > 26}$ ■

All the inequalities above have one condition. What if an inequality has two conditions? This would be a compound inequality. A compound inequality is shown below.

$$-3 < x \leq 1 \quad \text{is the same as} \quad \boxed{-3 < x} \text{ and } \boxed{x \leq 1}$$

This is basically saying that x is bounded by -3 as a lower boundary/limit, and 1 as an upper boundary/limit. Also, according to the inequality signs, it is noted that 3 is not included in the interval, while 1 is.

$-3 < x \leq 1$



Likewise, for the compound inequality, $-8 < 2x \leq 4$, there are two components or conditions. These are shown below.

$$-8 < 2x \leq 4 \quad \text{is the same as} \quad \boxed{-8 < 2x} \text{ and } \boxed{2x \leq 4}$$

As per solving the compound inequality, one must simply perfom the same usual steps to isolate the variable. The only additional thing is that these steps have to be applied to the three components of the equation. The left, middle, and right components.

$$-8 < 2x \leq 4 \qquad \frac{\cancel{2}(-4)}{\boxed{\cancel{2}}} < \frac{\cancel{2}(x)}{\boxed{\cancel{2}}} \leq \frac{\cancel{2}(2)}{\boxed{\cancel{2}}} \qquad \boxed{-4 < x \leq 2}$$

If division had been by a -2 then both inequality symbols would have been reversed.

For the compound inequality, $-10 < \dfrac{x}{-3} \leq 5$, there are two components or conditions. These are shown below.

$$-10 < \frac{x}{-3} \leq 5 \quad \text{is the same as} \quad \boxed{-10 < \frac{x}{-3}} \text{ and } \boxed{\frac{x}{-3} \leq 5}$$

The variable x is not alone. In order to isolate the variable x it is necessary to multiply by -3. This is shown below.

$$-10 < \frac{x}{-3} \leq 5 \qquad (-10)\boxed{\text{-3}} > \left(\frac{x}{\cancel{-3}}\right)\boxed{\cancel{-3}} \geq (5)\boxed{\text{-3}} \qquad 30 > x \geq -15$$

This last compound inequality would be rewritten as $\boxed{-15 \leq x < 30}$ for clarity. Recall that if multiplication or division occurs across an inequality symbol by a negative number, then the inequality symbol is reversed.

Solved Problem 2.4 Solve the following compound inequalities

(a) $48 \leq x + 50 < 60$ (b) $-14 < x - 10 \leq -6$

(a) $48 \leq x + 50 < 60 \quad 48\ \boxed{\text{-50}} \leq x + \cancel{50}\ \boxed{-\cancel{50}} < 60\ \boxed{\text{-50}}$

$\boxed{\text{Final Answer } -2 \leq x < 10}$

(b) $-14 < x - 10 \leq -6 \quad -14\ \boxed{+10} < x - \cancel{10}\ \boxed{+\cancel{10}} \leq -6\ \boxed{+10}$

$\boxed{\text{Final Answer } -4 < x \leq 4}$

Solved Problem 2.5 Solve the following compound inequalities

(a) $-100 < -2x < 8$ (b) $-3 < \dfrac{x}{-4} < 5$

(a) $-100 < -2x < 8 \quad \dfrac{\cancel{-2}(50)}{\boxed{\cancel{-2}}} > \dfrac{\cancel{-2}x}{\boxed{\cancel{-2}}} > \dfrac{\cancel{-2}(-4)}{\boxed{\cancel{-2}}} \quad 50 > x > -4$

The answer would be rewritten $\boxed{\text{Final Answer } -4 < x < 50}$

(b) $-3 < \dfrac{x}{-4} < 5 \quad (-3)\boxed{\text{-4}} > \left(\dfrac{x}{\cancel{-4}}\right)\boxed{\cancel{-4}} > (5)\boxed{\text{-4}} \quad 12 > x > -20$

The answer would be rewritten $\boxed{\text{Final Answer } -20 < x < 12}$ ▪

2.2 Sets and Intervals

Sets are used to group related elements together, facilitating operations like union and intersection. Intervals represent continuous ranges of values or time, essential for defining functions and expressing data succinctly. Both concepts streamline mathematical analysis and organization of information. Sets can be like a code, where a simple idea is communicated, in a different way.

A plain English message could be put inside curly brackets. $\boxed{\{\text{x is a natural number}\}}$

A second property could be added to this message, in plain English, in the following way:

$$\boxed{\{x \text{ is a natural number, } x \text{ is also greater than } 10\}}$$

Solved Problem 2.6 Recall the symbols used for real numbers. $\mathbb{R} \quad \mathbb{W} \quad \mathbb{Q} \quad \mathbb{Z} \quad \mathbb{N}$

1. The symbol for whole numbers is?
2. The symbol for rational numbers is?
3. The symbol for natural numbers is?
4. The symbol for integers is?

Final Answers: 1. $\mathbb{W}$ 2. $\mathbb{Q}$ 3. $\mathbb{N}$ 4. $\mathbb{Z}$ ▪

The $\{x \in \mathbb{N},\ x > 10\}$ message has 2 components.

The symbols $\boxed{x \in \mathbb{N}}$ mean:

- $\mathbb{N}$ means all real numbers
- x is an element of $\mathbb{N}$
- x is an element of all real numbers

The symbols $\boxed{x > 10}$ mean:

- x is greater than 10.

The previous message can be rewritten with an introduction or header in the following way:

$$\{x \mid x \in \mathbb{N},\ x > 10\}$$

The new symbols $\boxed{x \mid}$ mean:

- x such that x
- The set of elements x such that

This is **set-builder notation**. The symbols $\{x \mid x \in \mathbb{N},\ x > 10\}$ mean the set of elements x

such that $\boxed{x \text{ is an element of real numbers}}$ and $\boxed{x \text{ is greater than } 10}$. In other words, x is a real number, and x is greater than 10.

Solved Problem 2.7 Write the set of elements y, such that y is an integer, and $y < 1$, in **set-builder notation**.

Write the set of elements $\boxed{y}$, $\boxed{\text{such that } y \text{ is an}}$ $\boxed{\text{integer}}$, and $\boxed{y < 1}$, in set-builder notation?

While thinking in terms of components, one can see that the variable is y. $\boxed{\{y\,|\}}$

The next component adds one property of y, that it is an element of $\mathbb{Z}$, that it is an integer $\boxed{\{y \mid y \in \mathbb{Z}\}}$

The next component adds a second property of y $\boxed{\text{Final Answer} \quad \{y \mid y \in \mathbb{Z}, y < 1\}}$ ■

Solved Problem 2.8 Write the set of elements z, such that z is a whole number, and z is greater than 5 in **set-builder notation**.

The first component states that z is being described. It means z such that, then descriptions of z follow. So, the first component of the solution is $\boxed{\{z\,|\}}$

The next component adds one property of z $\boxed{\{z \mid z \in \mathbb{W}\}}$

Next add a second property of z $\boxed{\text{Final Answer} \quad \{z \mid z \in \mathbb{W},\ z > 5\}}$ ■

Solved Problem 2.9 Write the set of elements x, such that x is greater than or equal to -4 and less than or equal to 7, in **set-builder notation**

The first component states that x is being described. It means x such that, then descriptions of x follow. The first component of the solution would be $\boxed{\{x\,|\}}$

The next components add 2 properties of x $\boxed{\text{Final Answer} \quad \{x \mid\ x \geq -4,\ x \leq 7\}}$

This is rewritten in clearer, shorter way $\boxed{\text{Final Answer} \quad \{x \mid\ -4 \leq\ x\ \leq 7\}}$ ■

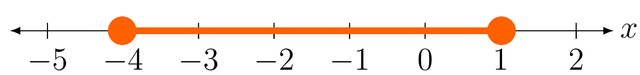
It is clear that $\{x \mid\ y > 5\}$, is referring to numbers greater than 5. **Interval notation** is another way to write this that is even shorter than set-builder notation.

$\{x \mid\ -4 \leq\ x\ \leq 1\}$ can also be written as [-4, 1], in **interval notation**. How so? Think in terms of components to understand. The expression [-4, 1] has 2 components $\boxed{[-4}$ and $\boxed{1]}$

- The $\boxed{[-4}$ means greater than or equal to -4
- The $\boxed{1]}$ means less than or equal to 1
- So, [-4, 1] means greater than or equal to -4 and less than or equal to 1

Definition 2.2 — Interval. An interval is simply a group or list of numbers between two endpoints. Each endpoint may or may not be included in the set or list of numbers.

Let's see the interval [-4, 1] in a visual way.

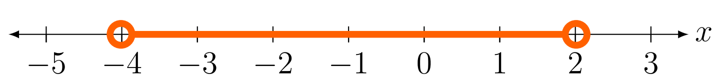


This interval spans from -4 to 1, and it has a solid dot at -4 and 1, this is because the interval includes both -4 and 1.

$\{x \mid -4 < x < 2\}$ can also be written as (-4, 2), in **interval notation**. Think in terms of components to understand this. The expression (-4, 2) has 2 components $\boxed{(-4}$, $\boxed{2)}$

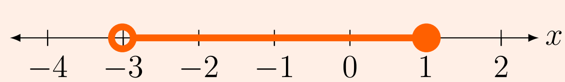
- The $\boxed{(-4}$ means greater than -4. It does not include -4.
- The $\boxed{2)}$ means less than 2. It does not include 2.
- So, (-4, 2) means greater than -4 and less than 2

Let's see (-4, 2) in a visual way.



This orange line spans from -4 to 2, and it has hollow circles at -4 and 2. This is because the interval does not include neither -4 nor 2.

Solved Problem 2.10 Describe the following interval in **interval notation**.

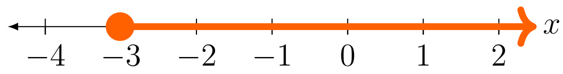


The interval above, shown in orange, has a starting point and an ending point. One can see that these two points are -3 and 1. It is also shown that:

- **-3 is not included** in the interval. In interval notation this is shown as $\boxed{(-3}$
- The point, **1 is included** in the interval. In interval notation this is shown as $\boxed{1]}$

These 2 components are assembled, with a comma. $\boxed{\text{Final Answer } (-3,1]}$ ■

How would the following interval be described?



In the interval above, one can see that:

- It starts at -3
- It includes -3
- It does not have an ending point
- It continues towards positive infinity

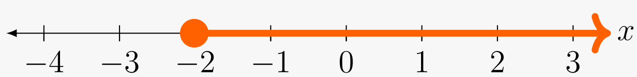
When a set of numbers continues towards positive infinity, the symbol $+\infty$ is used. When a set of numbers continues towards negative infinity, the symbol -∞ is used. This interval above would be written, $[-3,+\infty)$

Definition 2.3 — Infinity. Infinity describes the concept of endless, unlimited, or unbound. The symbol for infinity is ∞. On a real number line, there can be infinity in the positive direction $+\infty$, infinity in the negative direction $-\infty$, or both ∞. Infinity itself is not a real number.

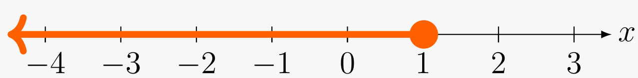
? In interval notation, why is a parenthesis used with either positive or negative ∞?

Tip A **parenthesis is used with $+\infty$ or -∞**, because infinity is not a real number, so it can not be included, in the interval.

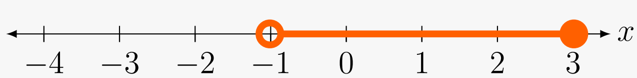
■ **Example 2.1** The following shows intervals and the corresponding interval notation.



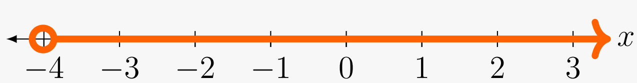
$[-2,+\infty)$



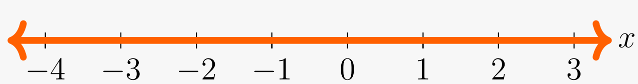
$(-\infty,1]$



$(1,3]$



(-1, $+\infty$)



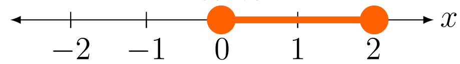
(-∞, $+\infty$)

■

? What if it is necessary to combine the following intervals?

Interval A

Interval B

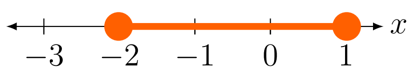


If one wants to "combine" Interval A and Interval B, shown above, it is necessary to be clear, about how they will be combined. Two possible routes would be to find the union or the intersection of Interval A and Interval B.

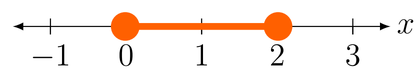
Definition 2.4 — Union. The union of two sets is the set of all elements in both sets. It can also be said that the union of two sets is the elements that are in the first set **OR** the second set. Likewise, the union of more than 2 sets is the elements in all the sets. **The symbol for union is** $\cup$

Definition 2.5 — Intersection. The intersection of two sets is the set of elements that is common to both sets. It can also be said that the intersection of two sets is the elements that are in the first set **AND** the second set. Likewise, the intersection of more than two sets is the elements common to all the sets. The symbol for intersection is $\cap$.

Interval A



Interval B

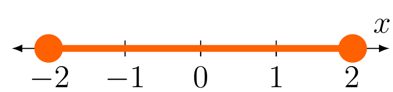


For Interval A, shown above, set notation would be [-2, 1]. Set-builder notation would be $\{x \mid -2 \leq x \leq 1\}$. Roster form would be {-2, -1, 0, 1}. **Roster form is just a list of the numbers that are included in the interval or set.**

For Interval B, shown above, set notation would be [0, 2]. Set-builder notation would be $\{x \mid 0 \leq x \leq 2\}$. Roster form would be {0, 1, 2}

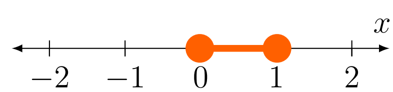
A $\cup$ B, The union of A and B:

- [-2, 2] in interval notation
- $\{x \mid -2 \leq x \leq 2\}$ in set-builder notation
- {-2, -1, 0, 1, 2} in roster form



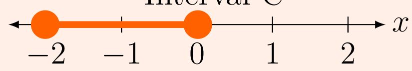
A $\cap$ B, The intersection of A and B:

- [0, 1] in interval notation
- $\{x \mid 0 \leq x \leq 1\}$ in set-builder notation
- {0, 1} in roster form

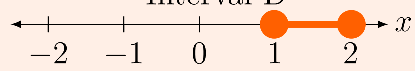


Solved Problem 2.11 Find C $\cap$ D.

Interval C



Interval D

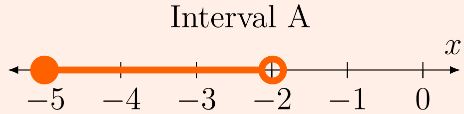


The two intervals above do not have any real number in common. The intersection would be an empty set, also called a null set. The empty set can be written as:

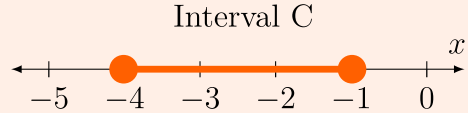
{}, empty curly brackets or ∅ (looks like zero) | Final Answer ∅ or {} |

Solved Problem 2.12 Find A ∩ C.

Interval A



Interval C

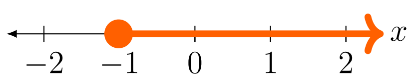


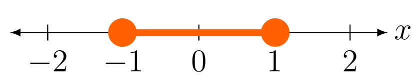
The two intervals above do have real numbers in common. The intersection would be {-4, -3} in roster form, $[-4, -2)$ in interval form, $\{x \mid -4 \leq x < -2\}$ in set-builder form

| Final Answer {-4, -3} or $[-4, -2)$ or $\{x \mid -4 \leq x < -2\}$ |

2.3 Graphing Inequalities

The number lines below illustrate both a simple inequality and a compound inequality. To the left one can see an inequality with one limit or threshold. To the right, one can see that the inequality is bounded by two limits, -1 and 1.





The same inequalities just above can be graphed on coordinate planes, in 2 dimensions, as shown, just below.

Fig. 2.1: $-1 \leq x$

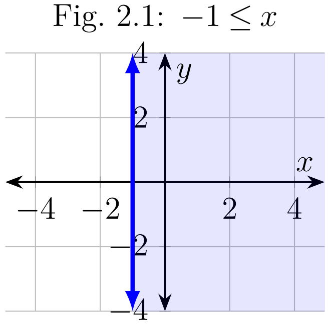
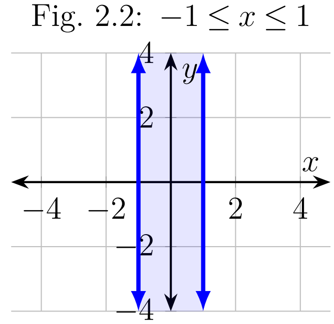


Fig. 2.2: $-1 \leq x \leq 1$



The graph of the inequality, in Figure 2.1 and 2.2, the boundaries $x = -1$ and $x = 1$ should be included The solid, blue, vertical line at $x = -1$ is a limit or boundary. This solid, vertical, blue line is included in the possible outputs.

In Figure 2.3 and 2.4, the boundaries are not included. For this reason, a dashed line is used, instead of a solid line.

Fig. 2.3: $-1 < x$

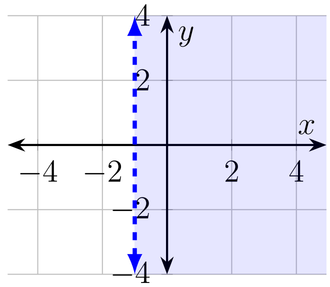
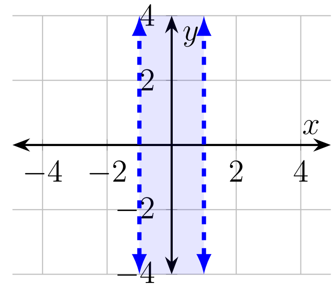


Fig. 2.4: $-1 < x < 1$



Recall that a compound inequality is the combination of simple inequalities.

In Figure 2.5, the boundary at $x = -2$ is not included, so this boundary is shown with a dashed line.

Fig. 2.5: $-2 < x \leq 4$

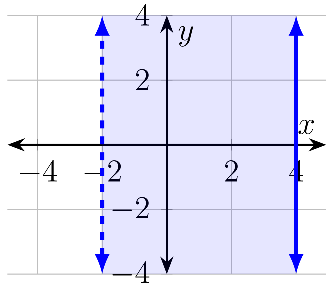


Fig. 2.6: $-3 < x \leq 1$

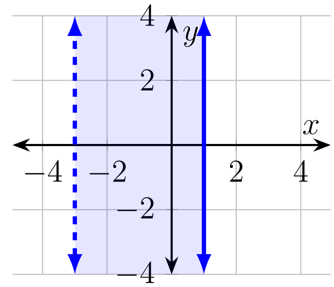


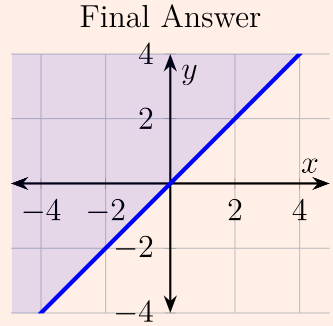
Figure 2.6, at $x = -3$ there is a dashed blue line because -3 is not included. At $x = 1$ there is a solid blue line because $x = 1$ is included.

Solved Problem 2.13 Graph the inequality $y \geq x$

The graph of the linear function $y = x$ should be familiar from Chapter 2.

The linear function $y = x$ is graphed as a solid blue line, because it is included in the possible outputs.

Final Answer



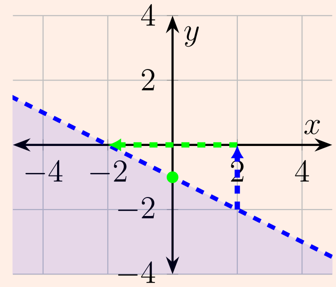
For this inequality $y \geq x$, the output is always greater than or equal to the input. In the graph, the solid blue line and the blue shaded region satisfy this condition. The output and solution, is any number greater than or equal to the input x. ■

Solved Problem 2.14 Graph the inequality $y < -\frac{x}{2} - 1$

The y-intercept of -1 is identified and plotted with a green dot.

The slope of the line should be $-\frac{1}{2}$.
A rise of 2 is shown with a blue, dashed arrow. A run of 4 is shown with a green, dashed arrow. Recall that $slope = \frac{rise}{run} = \frac{2}{-4} = -\frac{1}{2}$

Final Answer



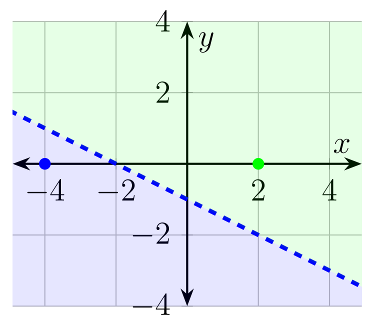
The inequality has a less than symbol, <. The linear function $y = -\frac{x}{2} - 1$ is graphed with a blue, dashed line, and this linear function is not included in the possible outputs.

If $x = 2$ then $y < -\frac{\boxed{2}}{2} - 1 \qquad y < \boxed{\text{-1 - 1}} \qquad y < -2$

This means that if $x = 2$ then the output is any number less than -2. ■

In this last solved problem a key step is to graph the boundary that is the linear function $y = -\frac{x}{2} - 1$. Then one needs to understand, which side of the line satisfies the condition of the inequality selecting points and test them in order to identify which side of the boundary is correct.

The point (2, 0) shown in green can be plugged into the inequality $y < -\frac{x}{2} - 1$ to test the green side of the boundary
The point (-4, 0) shown in blue can be plugged into the inequality $y < -\frac{x}{2} - 1$ to test the green side of the boundary



The green point (2, 0) is plugged into the inequality.

$\boxed{\text{y}} < -\frac{\boxed{\text{x}}}{2} - 1 \qquad \boxed{0} < -\frac{\boxed{2}}{2} - 1 \qquad 0 < -1 - 1 \qquad 0 < -2 \qquad \boxed{\text{This is false!}}$

The blue point (-4, 0) is plugged into the inequality.

$\boxed{\text{y}} < -\frac{\boxed{\text{x}}}{2} - 1 \qquad \boxed{0} < -\frac{\boxed{\text{-4}}}{2} - 1 \qquad 0 < 2 - 1 \qquad 0 < 1 \qquad \boxed{\text{This is true!}}$

Based on this test, the blue side of the line would be the correct side. The blue shaded region satisfies the inequality $y < -\frac{x}{2} - 1$. This test that was just demonstrated can also be useful with systems of inequalities. The example just above illustrates the inequality $y < -\frac{x}{2} - 1$. A second inequality can be added to form the following system of inequalities.

$y < -\frac{x}{2} - 1$ $\qquad$ $y > -2$

Fig. 2.7: $y < -\frac{x}{2} - 1$

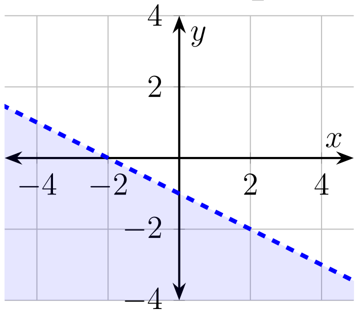
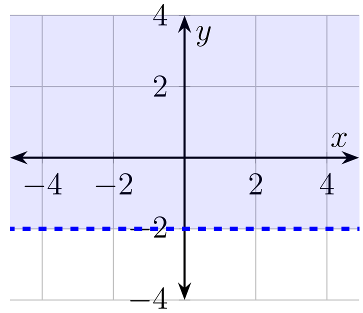


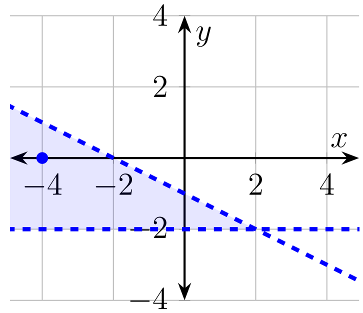
Fig. 2.8: $y > -2$



The graph in Figure 2.9 shows the intersection of the graphs in Figure 2.7 and Figure 2.8.

It is bound by two dashed lines because both inequalities made use of either a less than symbol, <, or a greater than symbol, >.

Fig. 2.9: System of Inequalities



In the graph in Figure 2.9, above, the blue point (-4, 0) is plugged into the inequality $y > -2$

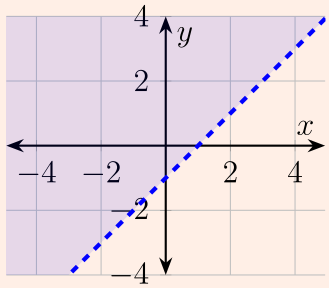
$y > -2 \qquad \boxed{0} > -2 \qquad \boxed{\text{This is true!}}$

The same blue point (-4, 0) is plugged into the inequality $y < -\frac{x}{2} - 1$.

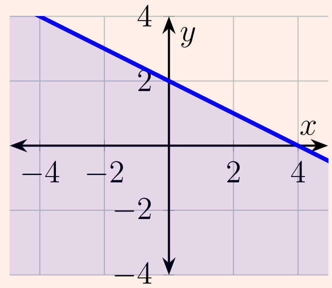
$\boxed{y} < -\frac{\boxed{x}}{2} - 1 \qquad \boxed{0} < -\frac{\boxed{-4}}{2} - 1 \qquad 0 < 2 - 1 \qquad 0 < 1 \qquad \boxed{\text{This is true!}}$

Solved Problem 2.15 Graph the system of inequalities $y > x - 1$ and $y \leq -\frac{x}{2} + 2$

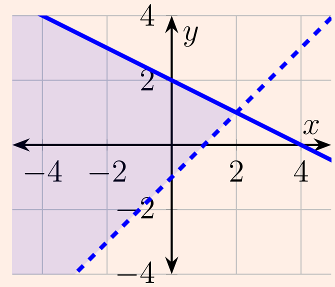
Graph D



Graph E



Graph F - Final Answer



Graph D, above, shows the graph for inequality $y > x - 1$. Graph E, below, shows the graph for inequality $y \leq -\dfrac{x}{2} + 2$.

Graph F, above, shows the intersection of Graph D and Graph E. The blue shaded region in Graph F satisfies the system of inequalities. ▪

2.4 Nonlinear Functions

The point has been emphasized that linear functions describe straight lines. Examples of this are shown below. The graphs in Figure 2.10 and Figure 2.11, below, are linear functions.

The key point about this linear function in Figure 2.10 is that the variable x has an exponent of 1.

Fig. 2.10: $f(x) = x$

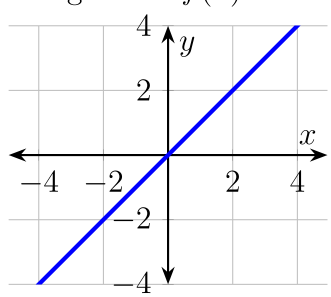
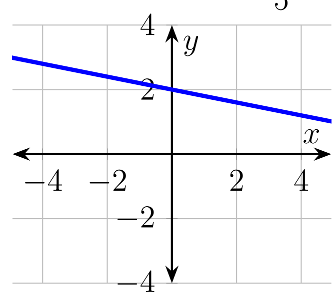
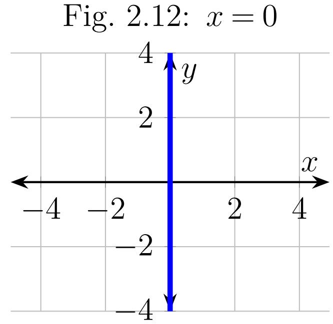


Fig. 2.11: $f(x) = -\dfrac{x}{5} + 2$



The graph in Figure 2.12, is not a function. The graph in Figure 2.12 is a straight line, but it will not pass the vertical line test. The vertical line test was discussed in Definition 1.3 in Chapter 1.

Fig. 2.12: $x = 0$



What happens if the linear function in Figure 2.10 is squared? In Figure 2.10, if the variable x is squared the function becomes $f(x) = x^2$. This is demonstrated in the graph in Figure 2.13, below.

If $x = -2$ $\boxed{f(x) = x^2} = (-2)^2 = 4$ So if $x = -2$, then $y = 4$ coordinates are $(-2, 4)$

If $x = -1$ $\boxed{f(x) = x^2} = (-1)^2 = 1$ So if $x = -1$, then $y = 1$ coordinates are $(-1, 1)$

If $x = 0$ $\boxed{f(x) = x^2} = 0^2 = 0$ So if $x = 0$, then $y = 0$ coordinates are $(0, 0)$

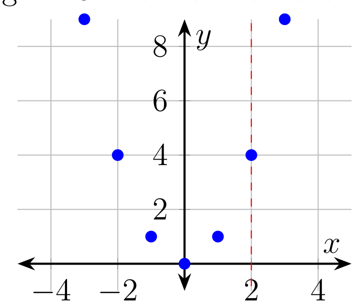
if $x = 2$ $\boxed{f(x) = x^2} = 2^2 = 4$ So if $x = 2$, then $y = 4$ coordinates are $(2, 4)$

The graph in Figure 2.13 shows seven points for the function $f(x) = x^2$

The highest exponent of x is 2, so this function has a **degree of 2**

This graph is not straight, so it is **not linear**

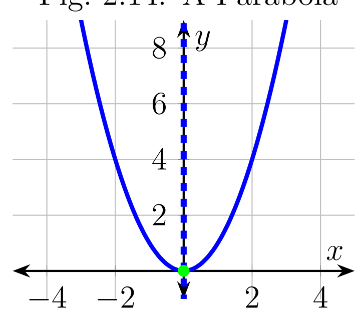
Fig. 2.13: Points in a Parabola



The graph in Figure 2.13, above, shows $f(x) = x^2$, which is a quadratic function. The red, dashed, vertical line only touches one point at a time. That makes $f(x) = x^2$ a **function. A key characteristic of a quadratic equation is that it includes x^2.**

The function $y = x^2$ graphed in Figure 2.14 has a special shape called a parabola. The axis of symmetry of the parabola, which divides it into 2 halves, is shown as a blue dashed line. Also, the vertex, which is the point where the parabola meets the axis of symmetry, is shown as a green dot.

Fig. 2.14: A Parabola



- $f(x) = x^2$ is the solid, blue, curved line
- It is not a straight line, not linear
- It is a function
- It is a nonlinear function
- The variable x has an exponent of 2
- $f(x) = x^2$ has a degree of 2
- The outputs $f(x)$ or y are always positive
- It is a parabola
- It has a shape like a "U"
- **It is a quadratic function**
- The green dot at $(0, 0)$ is called the vertex
- The blue, dashed line is the axis of symmetry
- The inputs x can be positive or negative

Definition 2.6 — Axis of Symmetry. An axis of symmetry is an imaginary line that divides a shape into two equal halves. At the axis of symmetry, two halves appear to be mirror images of each other.

In Figure 2.15, there is no way to split it in half. Also, the graph in Figure 2.16, does not display symmetry, and there is no way to divide it into two equal halves.

Fig. 2.15: No Symmetry

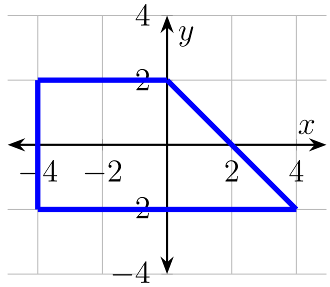
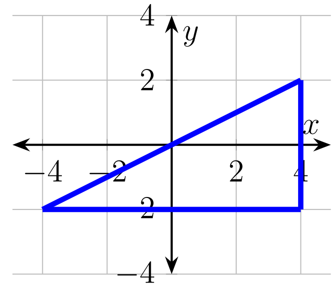
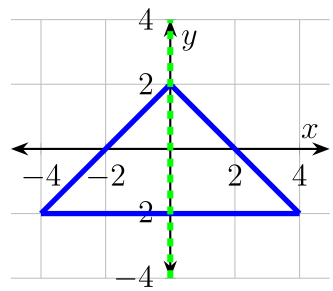


Fig. 2.16: No Symmetry



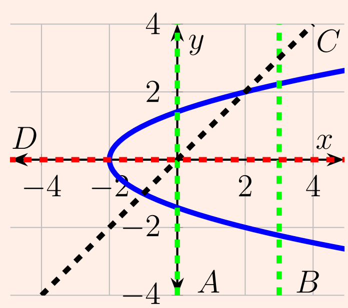
In the graph in Figure 2.17, it can be split in half by the axis of symmetry that is the vertical, dashed, green line.

Fig. 2.17: Symmetry



Solved Problem 2.16 For the blue parabola in the graph below, identify the axis of symmetry.

The green, dashed, vertical line A does not split the parabola into 2 halves. Same for line B, and C.



The blue line shows a sideways parabola that displays symmetry. Only the red, dashed, horizontal line D splits the parabola into 2 equal halves.

$\boxed{\text{Final Answer Red, dashed line D}}$

Definition 2.7 — Vertex of a Parabola. The vertex of a parabola is the point where the parabola and the axis of symmetry intersect.

Below, the graphs in Figures 2.18, 2.19, and 2.20, show parabolas in blue. the axis of symmetry is shown with a green, dashed line, and the vertex with a solid, red dot. In the graph in Figure 2.18, the vertex is at point (2, 2). In Figure 2.19, that the vertex is at point (-2, 0), and in Figure 2.20, the vertex is at point (-4, 0).

in Figure 2.20 is not a function. It is a parabola, but it is not a function since it fails the vertical line test.

Fig. 2.18: V = (2, 2)

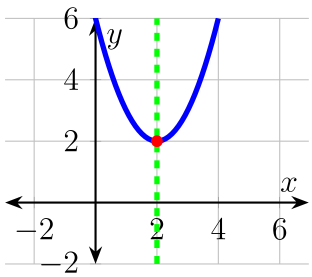


Fig. 2.19: V = (-2, 0)

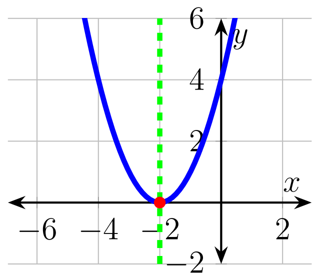
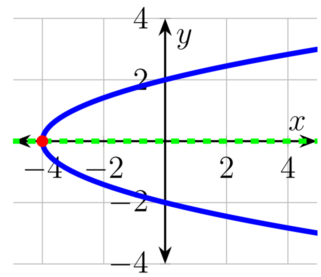


Fig. 2.20: V = (-4, 0)

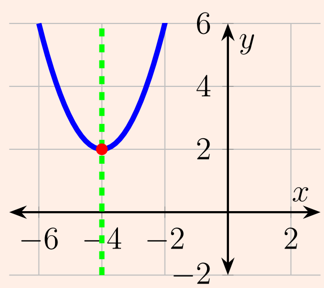


Solved Problem 2.17 For each parabola below, identify the vertex.

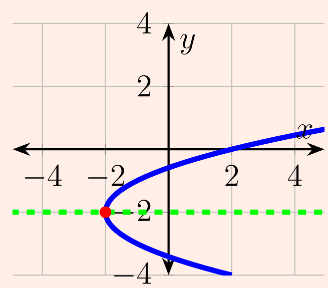
In Graph A the vertex is (-4, 2).

In Graph B the vertex is (-2, -2)

Graph A



Graph B



In both Graph A and Graph B, the vertex is shown with a red point. ■

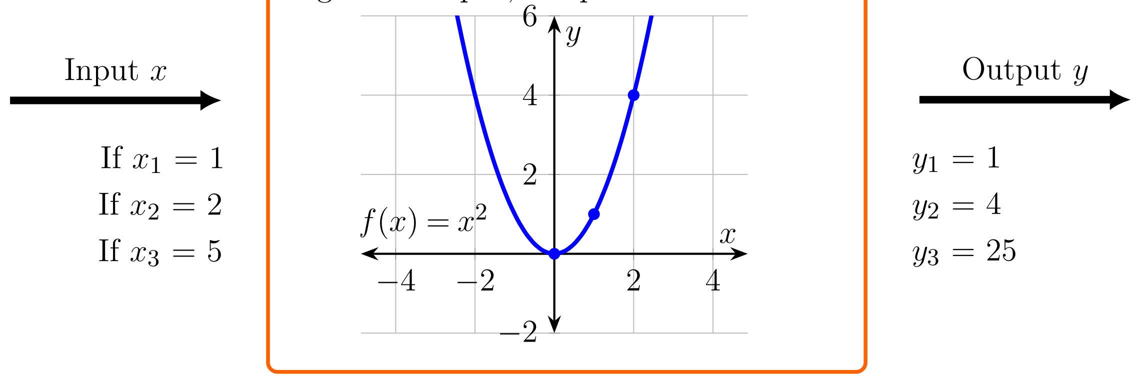
Definition 2.8 — Domain. The domain of a function is the set of all possible input values, or x values.

Definition 2.9 — Range. The range of a function is the set of all possible output values, or y values.

Figure 2.21 displays a parabola. It has the quadratic equation $f(x) = x^2$ and passes the vertical line test for a function, so this graph is a function. vertex is the a red dot. The x values continue in the positive direction in an endless manner. The input for $f(x) = x^2$ can be 1, 2, 5 or any positive number! The points (1, 1) and (2, 4) are shown below as blue dots.

If the input is 5 then the output is 25. This would lead to point (5, 25), which is not visible in the graph below, but it does exist.

Fig. 2.21: Input, Output in a Parabola



So for the function $f(x) = x^2$, any and all positive/negative values of x are allowed. The domain of the function $f(x) = x^2$, is all real numbers. In interval notation the domain of $f(x) = x^2$ is $(-\infty, +\infty)$.

In a similar manner, range describes the possible values for the output or the y value. In Figure 2.21, the possible values for y will vary from zero to infinity. In other words, the lowest possible value of y is zero and can extends to infinity. So, The range of the function would be $[0, \infty)$, in interval notation. In the three graphs, below, range is labeled as R.

Fig. 2.22: R: $[0, +\infty)$

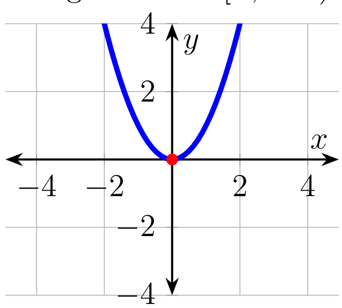


Fig. 2.23: R: $[2, +\infty)$

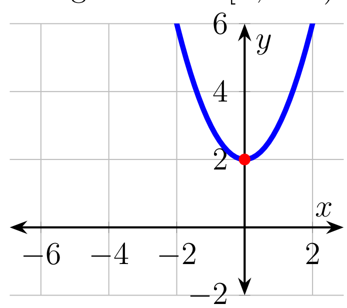
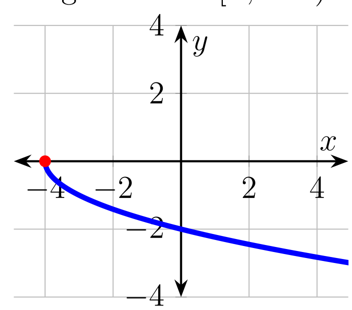


Fig. 2.24: R: $[0, -\infty)$



For the graph in Figure 2.22, above, the domain is $(-\infty, +\infty)$. The range is $[0, +\infty)$. In set-builder notation the domain would be $\{x \mid x \in \mathbb{R}\}$. The range would be $\{y \mid 0 \leq y\}$

For the graph in Figure 2.23, above, the domain is $(-\infty, +\infty)$. The range is $[2, +\infty)$. In set-builder notation the domain would be $\{x \mid x \in \mathbb{R}\}$. The range would be $\{y \mid 2 \leq y\}$.

For the graph in Figure 2.24, above, the domain is $[-4, +\infty)$. The range is $(-\infty, 0]$. In set-builder notation the domain would be $\{x \mid -4 \leq x\}$. The range would be $\{y \mid y \leq 0\}$

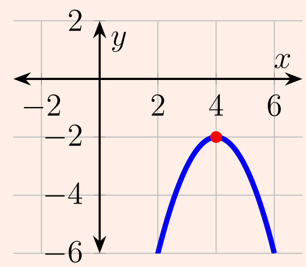
Solved Problem 2.18 Find the vertex, domain, and range of the nonlinear function, in the graph below.

Final Answers

The vertex is the point (4, -2)

The domain is
$(-\infty, +\infty)$ or $\{x \mid x \in \mathbb{R}\}$

The range is $(-\infty, -2]$ or $\{y \mid y \leq -2\}$



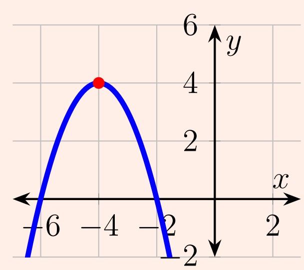
Solved Problem 2.19 Find the vertex, domain, and range of the linear function, in the graph below.

Final Answers

The vertex is the point (-4, 4)

The domain is
$(-\infty, +\infty)$ or $\{x \mid x \in \mathbb{R}\}$

The range is $(-\infty, 4]$ or $\{y \mid y \leq 4\}$



2.5 Quadratics

Quadratics are simply equations where a term is squared, such as $f(x) = x^2$. Other examples of quadratics are:

$$3x^2 + 5 \qquad 5x^2 + 3x + 2 \qquad -7x^2 - 4$$

Definition 2.10 — Quadratic Equation. A quadratic equation has one variable, and it has a degree of 2.

A quadratic will have the form:

$$\boxed{ax^2 + bx + c = 0}$$

Examples of quadratic equations are:

$$x^2 - 3x - 4 \qquad 3 + 2x + x^2$$
$$x^2 + 6 \qquad x^2$$

a, b, and c are coefficients, and x is the variable or
unknown. The coefficient b and/or c can be zero. **The coefficient a cannot be zero, so that x^2 is found in the equation.**

Tip A quadratic equation always has a variable that is squared.

Recall that for a linear equation the exponent on any variable is 1. In a quadratic, the exponent of the variable is 2. In other words, the degree is 2.

? What does it mean to solve a quadratic equation?

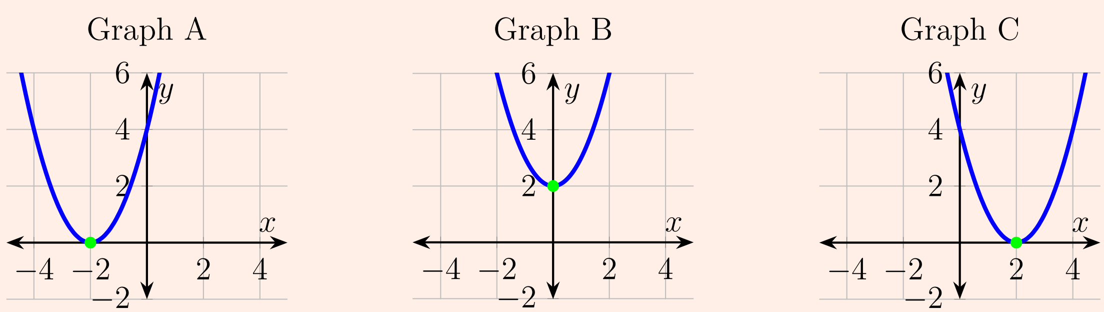
When one is solving a quadratic equation there can be:

- One solution
- Two solutions
- Zero real solutions

To solve any equation or function, one needs to find where the function touches the x-axis. Which is where the functions equals zero.

So, the standard form of a quadratic equation is $ax^2+bx+c=0$. This equation has three terms to the left of the equal sign. The term ax^2 is called a nonlinear term, because the variable x has an exponent of 2. The term bx is called a linear term. The term c is called a constant. If it is rewritten as $y=ax^2+bx+c$ or $f(x)=ax^2+bx+c$ then it is called a quadratic function.

Solved Problem 2.20 With a visual approach find the solution of the quadratic functions shown in the graphs below.



In Graph A, the function is equal to zero at $\boxed{\text{Final Answer } x=-2}$

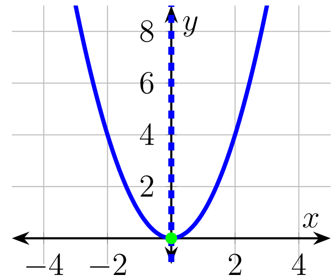
In Graph B, the function is never equal to zero $\boxed{\text{Final Answer is no solution}}$

In Graph C, the function is equal to zero at $\boxed{\text{Final Answer } x=2}$ ■

Tip Solutions of quadratic equations are also called zeros or roots

A quadratic function $f(x) = x^2$ is shown in Figure 2.25. It is a function called a parabola. The axis of symmetry is shown as a dashed, blue line. Here, there is one solution. It is equal to zero at only one point. That solution is the green dot found at $x = 0$.

Fig. 2.25: $f(x) = x^2$



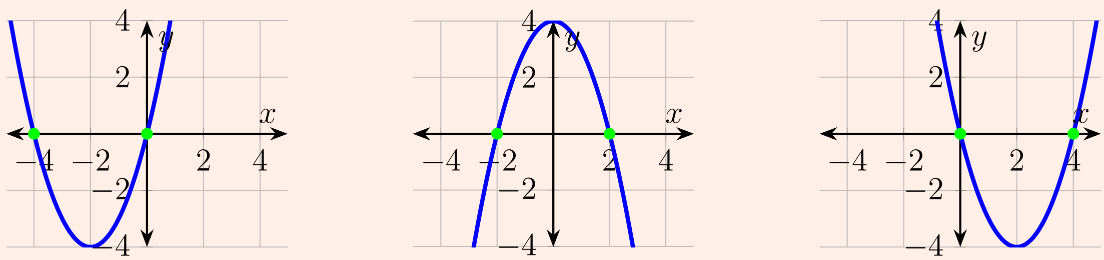
To solve a quadratic equation one needs to find points where $y = 0$. In Figure 2.25, $y = 0$ only at the vertex, which is (0, 0). So this function only one solution. If x^2 is made equal to zero, as in $x^2 = 0$ one could solve for x. In other words, what value of x would make $x^2 = 0$ true?

$$\sqrt{x^2} = \pm x \qquad \sqrt{x^2} = 0 \qquad x = 0 \text{ So the solution would be } 0$$

The equation x^2 is the simplest quadratic equation, and other quadratic equations build upon understanding of x^2. In the last solved problem in Graph B the graph never touches the x-axis. And in both Graph A and Graph C there is one solution. Often a quadratic function has 2 solutions.

Solved Problem 2.21 With a visual approach find the solutions of the quadratic functions shown in the graphs below.

Graph D Graph E Graph F



In Graph D, the function is equal to zero at $\boxed{\text{Final Answer } x = -4 \text{ and } x = 0}$

In Graph E, the function is equal to zero $\boxed{\text{Final Answer } x = -2 \text{ and } x = 2}$

In Graph F, the function is equal to zero at $\boxed{\text{Final Answer } x = 0 \text{ and } x = 4}$ ■

The last two solved problems show how quadratic equations can be solved quickly with a visual approach. This provides an understanding of the meaning of solutions to a quadratic equation. The previous section emphasized in a visual way, that solving a quadratic equation amounts to finding points where the graph intersects the x-axis. Solving a quadratic equation amounts to finding points where the equation, the output, or y is equal to zero.

Tip Isolate the unknown if its possible.

Tip Subtraction can be used to eliminate addition. Addition can be used to eliminate subtraction.

Tip Division can be used to eliminate multiplication. Multiplication can be used to eliminate division.

Tip Whatever change is made to one side of an equation, must also be applied to the other side of the equation.

Solving for an unknown is revisited here for simple quadratic equations, where the variable has an exponent of 2.

$$x^2 - 16 = 0 \qquad x^2 - 16 \boxed{+16} = 0 \boxed{+16} \qquad x^2 = 16$$

Tip There is a strong association between exponents and radicals. A radical can be used to eliminate an exponent. An exponent can be used to eliminate a radical.

Solved Problem 2.22 Solve for x in the following equations:

(a) $x^2 = 91$ (b) $x^2 = 144$ (c) $\sqrt{8} = \sqrt{x}$ (d) $\sqrt{11} = \sqrt{x}$

(a) There is a strong association between radicals and exponents. A square root can be used to eliminate an exponent of 2. $x^2 = 91 \quad \sqrt{x^2} = \sqrt{91} \quad \boxed{x = \pm 9}$

(b) $\sqrt{x^2} = \sqrt{144} \quad \boxed{x = \pm 12}$

(c) There is a strong association between radicals and exponents. An exponent of 2 can be used to eliminate a square root. $\sqrt{8} = \sqrt{x} \quad \left(\sqrt{8}\right)^2 = (\sqrt{x})^2 \quad \boxed{8 = x}$

Keep in mind that any value multiplied by itself is always positive. For this reason $\sqrt{11} \times \sqrt{11} = 11$

(d) $\sqrt{11} = \sqrt{x} \quad \left(\sqrt{11}\right)^2 = (\sqrt{x})^2 \quad \boxed{11 = x}$ ■

A square root applies a radical with an index of 2 which makes it possible to eliminate an exponent of 2. This is shown here below.

$$x^2 = 16 \qquad \sqrt{x^2} = \sqrt{16} \qquad \boxed{x = \pm 4}$$

Just above, a radical/square root was applied to both sides of the equation to eliminate the exponent of 2. The index 2 of this radical expression is even. For this reason, the solution can be either positive or negative. In other words, the solution is either +4 or -4.

The steps in reverse-PEMDAS, are reinforced below.

S - Subtraction	If subtraction exists, then eliminate it.
A - Addition	If addition exists, then eliminate it.
D - Division	If division exists, then eliminate it.
M - Multiplication	If multiplication exists, then eliminate it.
E - Exponents	If an exponent exists, then eliminate it.
P - Parentheses	If parentheses exist, then eliminate them.

Solving for an unknown is demonstrated again below.

$$x^2 - 100 = 0 \qquad x^2 - 100 \boxed{+100} = 0 \boxed{+100} \qquad x^2 = 100$$

$$x^2 = 100 \qquad \sqrt{x^2} = \sqrt{100} \qquad \boxed{x = \pm 10}$$

In this example, above, in order to isolate the variable x it was only necessary to:

- Eliminate subtraction
- Eliminate the exponent of 2

If reverse-PEMDAS is applied the first necessary step is, if subtraction exists, eliminate it. The only second necessary step is, if an exponent exists, eliminate it.

Solved Problem 2.23 Solve for x in the following quadratic equations:

(a) $\dfrac{x^2}{2} - 18 = 0$ (b) $2x^2 - 32 = 0$

(a) $\dfrac{x^2}{2} - 18 = 0$ A first step is to eliminate subtraction by adding 18 to both sides. The next step is to eliminate division by multiplying both sides by 2. The last step is to eliminate the exponent by applying a square root to both sides.

$$\frac{x^2}{2} - 18 + 18 = 0 + 18 \qquad \frac{x^2}{2} = 18 \qquad \frac{\cancel{2}x^2}{\cancel{2}} = 2(18) \qquad x^2 = 36 \quad \sqrt{x^2} = \sqrt{36} \qquad \boxed{x = \pm 6}$$

(b) $2x^2 - 32 = 0$ A first step is to eliminate subtraction by adding 32 to both sides. The next step is to eliminate multiplication by dividing both sides by 2. The last step is eliminate the exponent by applying a square root to both sides.

$$2x^2 - 32 + 32 = 0 + 32 \qquad 2x^2 = 32 \qquad \frac{\cancel{2}x^2}{\cancel{2}} = \frac{\cancel{2}16}{\cancel{2}} \qquad x^2 = 16 \quad \sqrt{x^2} = \sqrt{16} \qquad \boxed{x = \pm 4}$$

The quadratics discussed in this subsection only have one term with the variable x. When a quadratic equation has two terms with the variable x, then isolating the variable won't do any good. Additional methods may be necessary to solve the quadratic equation. Recall that solving a quadratic equation means finding points where it intersects the x-axis

In the expression $\sqrt{81} = \pm 9$, notice that 81 has factors of 9 and -9. These factors are solutions or square roots because $9 \times 9 = 81$ and $-9 \times -9 = 81$. Similarly, when dealing with quadratics, the solutions are called roots. Finding factors can aid in solving quadratics just like in the simple expression $\sqrt{81} = \pm 9$.

One needs to be comfortable, with multiplying binomials, first. If one practices multiplying binomials well, then solving quadratics is a lot simpler.

Tip Multiplying binomials is simply multiplying in an orderly way!

Multiplying binomials can be approached in three different ways:

- The distributive property
- The FOIL method
- The Grid method

All three methods give the same result. They all come down to the distributive property. While the FOIL and Grid methods offer visual aids, they're essentially variations of the distributive property. Let's explore this further with a solved problem using the distributive property

Solved Problem 2.24 Evaluate the following expressions by multiplying binomials.

(a) $(2x+5)(2x+5)$ (b) $(x-2)(x-2)$ (c) $(3x-4)(3x-2)$

(a) $(2x+5)(2x+5) = 4x^2 + 10x + 10x + 25 = \boxed{4x^2 + 20x + 25}$

(b) $(x-2)(x-2) = x^2 - 2x - 2x + 4 = \boxed{x^2 - 4x + 4}$

(c) $(3x-4)(3x-2) = 9x^2 - 12x - 6x + 8 = \boxed{9x^2 - 18x + 8}$

Recall that factors are multiplied and terms are added.

In the expression $(3x-4)(3x-2) = 9x^2 - 18x + 16$,

both $\boxed{(3x-4)}$ and $\boxed{(3x-2)}$ are factors of the equation $9x^2 - 18x + 8$.

- **F**irst
- **O**uter
- **I**nner
- **L**ast

The last exercise in the last solved problem is rewritten below. Two binomials are being multiplied. Each one of the 4 curved red and blue arrows below, corresponds to each letter in FOIL.

$$(3x-4)(3x-2) = 9x^2 - 12x - 6x + 8 = \boxed{9x^2 - 18x + 8}$$

In this example, just above:

- $\boxed{\textbf{F}}$ irst - corresponds to the red, curved arrow number 1
- $\boxed{\textbf{O}}$ uter - corresponds to the red, curved arrow number 2
- $\boxed{\textbf{I}}$ nner - corresponds to the blue, curved arrow number 3
- $\boxed{\textbf{L}}$ ast - corresponds to the blue, curved arrow number 4

In the explanation above, the red curved arrow labeled 2 demonstrates the multiplication of the two outermost terms, $3x$ and $-4y$, representing the "O" in FOIL. Similarly, the blue curved arrow labeled 3 shows the multiplication of the two innermost terms, $-4y$ and $3x$, representing the "I" in FOIL.

Below in Table 2.1, the grid method is depicted. Here, the terms of the first binomial occupy the first orange row, while the terms of the second binomial are placed in the first orange column. Let's explore the grid method further.

Table 2.1: Grid Method $(3x-4)(3x-2)$

Grid Method	$3x$	-4
$3x$	$3x \times 3x = 9x^2$	$3x \times -4 = -12x$
-2	$-2 \times 3x = -6x$	$-2 \times -4 = 8$

The results in the 4 white cells are then combined to form the answer.

$$9x^2 - 12x - 6x + 8 = \boxed{9x^2 - 18x + 8}$$

Table 2.2: Grid Method $(2x-5)(3x-2)$

Grid Method	$2x$	-5
$3x$	$3x \times 2x = 6x^2$	$3x \times -5 = -15x$
-2	$-2 \times 2x = -4x$	$-2 \times -5 = 10$

The results in the 4 white cells are then combined to form the answer.

$$\boxed{6x^2} - \boxed{15x} - \boxed{4x} + \boxed{10} = \boxed{6x^2 - 19x + 10}$$

Both the distributive method and the grid method deliver the same result.

$$(2x-5)(3x-2) = 6x^2 - 15x - 4x + 10 = \boxed{6x^2 - 19x + 10}$$

If you're given the expression $\boxed{5x+5}$, you might need to factor it. Factoring involves finding the components that, when multiplied, form the initial expression. It's essentially the reverse of applying the distributive property

In factoring one literally needs to think about what component or piece is found in all the terms. To factor, the first step is to find what is common to all the terms. In the expression $\boxed{5x}+\boxed{5}$, 5 shows up in both terms. Since 5 shows up in all terms it is clearly a factor. If 5 is then removed from both terms, only $x+1$ is left.

$$5x+5 = 5(\boxed{x}+\boxed{1}) = 5(x+1) \qquad \text{The factors are } \boxed{5} \text{ and } \boxed{x+1}$$

Tip When factoring it is possible to test or check the answers!

To check if factors are correct, apply distribution to the factors. If this leads to the initial expression then the factors are correct.

Solved Problem 2.25 Greatest common factor - Factor the expression $7x+7y+7z$.

To factor, it is necessary to find what component is found in ALL the terms.

$7x+7y+7z$	This is the **initial expression** that will be factored
$\boxed{7x}+\boxed{7y}+\boxed{7z}$	7 is found in ALL the terms, so it is clearly a factor
$7(\boxed{x}+\boxed{y}+\boxed{z})$	Pull 7 out of each term and multiply by what is left

$\boxed{\text{Final Answer} \quad \text{The factors are} \quad 7(x+y+z)}$ The factors can be tested!

$$(7)(x+y+z) = 7x+7y+7z$$

Distribution leads to the initial expression, so our factors are correct! ■

The solved problem, above, is quite simple, but with a similar approach, it will be possible to factor more interesting expressions. In the solved problem below, the two terms have something in common. The variables are x, y, and z. The variables are not really the focus, the focus here is to see what is common and what is different in the 2 terms.

Solved Problem 2.26 **Greatest common factor** - Factor the expression $7xy^2z + 7xy^2k$.

Again, it is necessary to find what component or piece is found in ALL the terms. This equation has two terms, but both terms have something in common.

$$\boxed{7xy^2}\, z + \boxed{7xy^2}\, k$$

$7xy^2(\boxed{z} + \boxed{k})$ Pull $7xy^2$ out, and multiply by what is left

$\boxed{\text{Final Answer The factors are } 7xy^2(z+k)}$ The factors can be tested!

$$(7xy^2)(z+k) = 7xy^2z + 7xy^2k$$

1 2

Distribution leads to the initial expression, so our factors are correct!

$$7xy^2z + 7xy^2k \qquad \boxed{7}\boxed{x}\boxed{y^2}\boxed{z} + \boxed{7}\boxed{x}\boxed{y^2}\boxed{k}$$

The expression above has 2 terms. On the right an orange border shows different components of the expression. The following observations can be made about this expression:

- $\boxed{\text{The constant or coefficient 7 is found in both terms}}$
- $\boxed{x \text{ is found in both terms}}$, $\boxed{y^2 \text{ is found in both terms}}$
- Since $7xy^2$ is found in both terms, it is a factor of the expression
- The factors of the expression are $7xy^2$ and $(z+k)$

Now consider the first term above. What if 7 is "hidden" inside another constant? Recall that $14 = 7 \times 2$ so 14 has a 7 "hidden" inside it. The expression above could be changed into the following new expression.

$$14xy^2z + 7xy^2k \qquad \boxed{14}\, xy^2z + \boxed{7}\, xy^2k$$

Is 7 still found in both terms? Sure, it is. Recall that $14 = 7 \times 2$. So, 7 is "hidden", inside 14. In order words, the **greatest common factor** between 14 and 7 is 7.

$$\boxed{2 \times 7}\, xy^2z + \boxed{7}\, xy^2k \qquad 2\times\boxed{7xy^2}\, z + \boxed{7xy^2}\, k \qquad \boxed{7xy^2}(2z) + \boxed{7xy^2}(k)$$

So, the factors are $7xy^2\ (2z+k)$.

z, k, and 2 are not found in **both** terms. We basically can't factorize further.

$$4xy^3z + 4xy^2k \qquad \boxed{4}\boxed{x}\boxed{y^3}\boxed{z} + \boxed{4}\boxed{x}\boxed{y^2}\boxed{k}$$

In the expression just above, the first term should be considered. What if y^2 is hidden inside another component. Recall that $y^3 = y \times y \times y$ and $y^3 = y^2 \times y$. So, y^2 is hidden inside y^3. The expression above could be changed to:

$$4xy^3z + xy^2k \qquad 4x\boxed{y^3}z + 4x\boxed{y^2}k$$

Is y^2 still found in both terms? Sure, it is. Recall that $y^3 = y^2 \times y$. So y^2 is hidden inside y^3. The greatest common factor of y^3 and y^2 is y^2.

$$4x\boxed{y^2(y)}z + 4x\boxed{y^2}k \qquad \boxed{4xy^2}yz + \boxed{4xy^2}k \qquad \text{Factors are: } 4xy^2\ (yz+k)$$

Solved Problem 2.27 Greatest common factor - Factor $28x^3y^2 + 14x^2y^4 + 7xy^5$.

This expression can be seen in terms of components and some observations can be made.

$$\boxed{28}\boxed{x^3}\boxed{y^2} + \boxed{14}\boxed{x^2}\boxed{y^4} + \boxed{7}\boxed{x}\boxed{y^5}$$

- This expression has 3 terms, separated by the plus sign
- The constant components are 28, 14, and 7
- The variable x shows up in the components x^3, x^2, and x
- The variable y shows up in the components y^2, y^4, and y^5

With these components in mind one needs to find what is hidden, and what shows up in all 3 terms. This refers to the greatest common factors.

The factors of 28 are 1, 2, 4, $\boxed{7}$, 28 The factors of 14 are 1, 2, $\boxed{7}$, 14

28, 14, 7 The greatest common factor is $\boxed{7}$ $28 = 4 \times 7, 14 = 2 \times 7$

x^3, x^2, x The greatest common factor is $\boxed{x}$ $x^3 = x^2 \times x, x^2 = x \times x$

y^2, y^4, y^5 Greatest common factor is $\boxed{y^2}$ $y^4 = y^2 \times y^2, y^5 = y^2 \times y^3$

Below, hidden information is shown inside each component. This makes it possible to see the greatest common factors. Notice, **there is an orange dot above every greatest common factor**.

$$\boxed{4 \times \dot{7}}\boxed{x \times \dot{x}^2}\boxed{\dot{y}^2} + \boxed{2 \times \dot{7}}\boxed{\dot{x} \times x}\boxed{\dot{y}^2 \times y^4} + \boxed{\dot{7}}\boxed{\dot{x}}\boxed{\dot{y}^2 \times y^3}$$

In each term, the greatest common factors are pulled out, and what is left is shown here with an orange border $7xy^2\ \boxed{4x^2} + 7xy^2\ \boxed{2xy^2} + 7xy^2\ \boxed{y^3}$

The component that shows up in all three terms is now pulled out, and this leads to the factors. $\boxed{\text{Final Answer} \quad \text{The factors are } 7xy^2(4x^2+2xy^2+y^3)}$ ▪

Solving a quadratic equation involves finding factors. One can find factors that were multiplied to form the quadratic equation. Each factor is then set equal to zero to find the solutions. The following are additional methods of solving quadratic equations.

- Sign recognition
- AC method
- Factoring formulas

Factoring a quadratic equation simplifies it by finding its component parts through multiplication. The three main methods are sign recognition, the AC method, and factoring formulas. Sign recognition and the AC method may involve trial and error, while factoring formulas require identifying the correct formula and applying it.

Solved Problem 2.28 Given the standard form of a quadratic equation $ax^2+bx+c=0$ find the coefficients a, b, and c in the equation $x^2+8x+12=0$.

The coefficients are shown with an orange border. $\boxed{1}\ x^2+\boxed{8}\ x+\boxed{12}=0$

In the term $\boxed{x^2}$, it may seem that the coefficient is missing, but the coefficient is in fact one. Recall that $3=1(3) \quad x=1(x) \quad x^2=1(x^2)$

$\boxed{\text{Final Answers: The coefficient } a \text{ is 1. Then } b \text{ is 8. Then } c \text{ is 12.}}$ ▪

If one needs to solve a quadratic equation a good starting point is the sign recognition method. Three steps help while learning and practicing the sign recognition method.

Step 1 Identify the correct signs of the binomial factors
Step 2 Identify the factors of c that add up to b
Step 3 Solve each binomial factor, for x

In Solved Problem 2.20, the quadratic equation x^2-4x+4 was formed by multiplying $(x-2)$ and $(x-2)$. To backtrack and find these binomials from the quadratic equation, note that the second term is subtracted and the third term is added. The coefficients of the quadratic equation are 1, -4, and 4 respectively, allowing us to find the binomials. Since the binomials must include x , there are four possible combinations.

Sign recognition step 1: Identify the correct signs of the binomial factors

In $x^2-4x+4 \qquad a=1\ \boxed{b=-4}\ \boxed{c=4} \qquad b$ and c are clearly positive

$\boxed{1.\ (x+\ \ )(x+\ \ )}$ 2. $(x-\ \ )(x+\ \ )$ 3. $(x+\ \ )(x-\ \ )$ $\boxed{4.\ (x-\ \ )(x-\ \ )}$

One can apply the distributive property to all 4 possibilities above, but only 2 of these possibilities will lead to a positive coefficient c.

1. $(x+\ )(x+\ )$ and 4. $(x-\ )(x-\ )$ lead to a positive coefficient c. A positive multiplied by a positive equals a positive, example $(+1)\times(+1)=1$. A negative multiplied by a negative equals a positive, example $(-1)\times(-1)=1$. The number of possibilities was narrowed from 4 possibilities to 2 possibilities. Now, which of these 2 possibilities leads to a negative coefficient, $-b$ when added?

In 1. $(x-\ )(x-\ )$ a negative added to a negative produces a negative, **so this possibility is correct**. In the other possibility a positive added to a positive must produce a positive.

Sign recognition step 2: Identify the factors of c that add up to b

The solution will have binomials in the form $(x-\ )(x-\ )$. Now one needs 2 numbers that multiply to produce 4, and these numbers must also add to make -4. In the quadratic equation $x^2-4x+\boxed{4}$, $c=4$, and c will have a few sets of factors, so these can be tested.

1 and 4 $\boxed{\text{2 and 2}}$ -2 and -2

Notice the signs in the form $(x-\ )(x-\ )$ where one can see 2 subtraction symbols or negatives! So the factors above are tested with $-\boxed{}-\boxed{}=-4$ to see if they add or subtract to make -4!

> It helps a lot to emphasize this point. The factors -2 and -2 when added as in $-2+(-2)=-2-2=-4$ do equal -4, BUT THESE ARE THE WRONG FACTORS!. With this problem one must use $-\boxed{}-\boxed{}=-4$ to test the factors, only because the chosen binomial form $(x-\ )(x-\ )$ has 2 subtraction or 2 negative symbols.

The factors 1 and 4 do multiply to make 4, but $-\boxed{1}-\boxed{4}=-5$ is not equal to the coefficient $c=-4$.

The factors 2 and 2 do multiply to make 4. Then $-\boxed{2}-\boxed{2}=-4$ is equal to the coefficient $c=-4$. **The factors 2 and 2 are the correct numbers to complete the binomials**.

The factors -2 and -2 do multiply to make 4. **BUT** $-\boxed{(-2)}-\boxed{(-2)}=4+4=8$ **is not equal to the coefficient** $c=-4$.

The form $(x-\boxed{})(x-\boxed{})$, applies to this problem. The factors 2 and 2 are plugged in to complete the binomials $(x-2)(x-2)$. These are factors of the original quadratic equation. Without the graph in the visual approach how can one find the zeros, roots, or solutions?

Sign recognition step 3: Solve each binomial factor for x

One sets each binomial equal to zero in order to solve for x. For this problem both binomials are the same, so really there is only one binomial. Example: $x-2=0$ then add 2 to both sides to solve for x. $x-\cancel{2}+\cancel{2}=0+2$. So $\boxed{x=2}$ is a solution of the quadratic equation x^2-4x+4. This means that at the point where $x=2$ the quadratic equation is equal to zero.

Solved Problem 2.29 Solve $x^2+10x+21=0$ by way of the sign recognition method.

In the quadratic equation $x^2+\boxed{\text{10x}}+\boxed{21}$ $\quad b=10$ $c=21$

Sign recognition step 1: Identify the correct signs of the binomial factors

The factors will have the form $(x\ \ )(x\ \ )$. The 4 possible options are:

$\boxed{1.\ (x+\ \ )(x+\ \ )}$ $\quad$ 2. $(x-\ \ )(x+\ \ )$ $\quad$ 3. $(x+\ \ )(x-\ \ )$ $\quad$ $\boxed{4.\ (x-\ \ )(x-\ \ )}$

The coefficient c is positive and only option 1 and 4 will produce a positive coefficient c.

The coefficient b is also positive and out of the options 1 and 4, only option 1 will also produce a positive coefficient b. The correct form for the binomial factors is $(x+\ \ )(x+\ \ )$

Sign recognition step 2: Identify the factors of c that add up to b

Then one identifies factors that multiply to produce $c=21$. These factors must also add up to $b=10$. Factor pairs of 21 are listed, 1 and 21 $\quad$ $\boxed{\text{3 and 7}}$.

Since the binomial form $(x+\ \ )(x+\ \ )$ was chosen. The possible factors are tested with $+\boxed{}+\boxed{}=10$. Only the factors 3 and 7 add up to 10, $+\boxed{3}+\boxed{7}=10$. So these numbers complete the binomial factors $\boxed{(x+3)(x+7)}$.

Sign recognition step 3: Solve each binomial factor for x

In the last step, set each binomial equal to zero, to solve for x. $\boxed{x+3=0}$ is rewritten as $x+\cancel{3}-\cancel{3}=-3$ so $x=-3$. Then $x+7=0$ is rewritten as $x+\cancel{7}-\cancel{7}=0-\cancel{7}$ so $\boxed{x=-7}$

$\boxed{\text{Final Answer } x=-3 \text{ and } x=-7}$ ▪

IMPORTANT: The sign recognition method can only be used when the coefficient of x^2 is $a=1$. Which is called **the leading coefficient.**

Solved Problem 2.30 Solve $x^2+8x+12=0$ by way of the sign recognition method.

In the quadratic equation $x^2+\boxed{\text{8x}}+\boxed{12}$ $\quad a=1$ $b=8$ $c=12$

Sign recognition step 1: Identify the correct signs of the binomial factors

The factors will have the form $(x \quad)(x \quad)$. The 4 possible options are:

$\boxed{1.\ (x+\ )(x+\ )}$ 2. $(x-\ )(x+\ )$ 3. $(x+\ )(x-\ )$ $\boxed{4.\ (x-\ )(x-\ )}$

The coefficient c is positive and only option 1 and 4 will produce a positive coefficient c.

The coefficient b is also positive and out of the options 1 and 4, only option 1 will also produce a positive coefficient b. The correct form for the binomial factors is $(x+\ )(x+\ )$

Sign recognition step 2: Identify the factors of c that add up to b

Then one identifies factors that multiply to produce $c = 12$. These factors must also add up to $b = 8$. Factor pairs of 12 are listed.

1 and 12 $\boxed{\text{2 and 6}}$ 3 and 4

Since the binomial form $(x+\ )(x+\ )$ was chosen. The possible factors are tested with $+\boxed{}+\boxed{} = 8$ Only the factors 2 and 6 add up to 8. So these numbers complete the binomial factors $\boxed{(x+2)(x+6)}$.

Sign recognition step 3: Solve each binomial factor for x

In the last step, set each binomial equal to zero to solve for x. $\boxed{x+2=0}$ is rewritten as $x+\cancel{2}-\cancel{2} = -2$ so $x = -2$. Then $x+6=0$ is rewritten as $x+\cancel{6}-\cancel{6} = 0-\cancel{6}$ so $\boxed{x=-7}$

$\boxed{\text{Final Answer } x = -2 \text{ and } x = -6}$ ■

For cases where a is not equal to 1, one can make use of the AC method. Five steps help while practicing and learning the AC method.

Step 1 Identify factors of $a \times c$ that add up to b
Step 2 Rewrite the quadratic equation with 4 terms
Step 3 Group the terms and factor the 2 groups
Step 4 Factor once more
Step 5 Solve each binomial factor for x

AC Method Step 1 Identify factors of $a \times c$ that add up to b

The table below, has the product $a \times c$ in the left column. Then b is in the right column. In the left column one can see rows with factors. These are factors of the product $a \times c$. In other words, these are factors of 45. Then one can check if these factors add up to b in the right column. One can see that the factors -5 and 9, in the last row add up to -14. Again, the objective is to find 2 factors of 45 that add up to 14.

Table 2.3: AC Method $3x^2 - 14x + 15$ $a = 3$ $c = 15$ $b = 14$

$ac = 3 \times 15 = 45$	$b = 14$
1 and 45	$1 + 45 = 46$
-1 and -45	$-1 - 45 = -46$
3 and 15	$3 + 15 = 18$
-3 and -15	$-3 - 15 = -18$
5 and 9	$5 + 9 = 14$
$\boxed{-5}$ and $\boxed{-9}$	$-5 - 9 = \boxed{-14}$

AC Method Step 2 Rewrite the quadratic equation with 4 terms

In this last table, one can see that factors of 45 that add up to -14 have been identified. Now the original quadratic equation is rewritten with four terms, by making use of these factors, $3x^2 - 9x - 5x + 15$. Notice that $9x$ has more in common with $3x^2$ than 15, for example 3 is a factor of 9. Notice that $5x$ has more in common with 15 than with $3x^2$, for example 5 is a factor of 15.

AC Method Step 3 Group the terms and factor the 2 groups

A next step is to group and factor terms. In the equation $3x^2 - 9x - 5x \boxed{+} 15$ terms can be grouped in this way $(3x^2 - 9x) - (5x \boxed{-} 15)$. **Notice the subtraction symbol between the two groups!** That the last addition symbol changed to a subtraction symbol can seem confusing at first! Recall that $-1(5x - 15) = -1(5x) - 1(15) = -5x + 15$. So the last addition symbol had to change to a subtraction symbol due to adding the groups with parentheses.

Each group can then be factored further by finding the greatest common factor in each.

The group $3x^2 - 9x$ is the same as $\boxed{\text{3x}}(x) - \boxed{\text{3x}}(3)$. Pull out the greatest common factor $3x$ to make $\boxed{3x(x-3)}$.

The group $5x - 15$ is the same as $5(x) - 5(3)$. In $\boxed{5}(x) - \boxed{5}(3)$ pull out the greatest common factor 5 to make $\boxed{5(x-3)}$

AC Method Step 4 Factor once more

When the groups $\boxed{3x(x-3)}$ and $\boxed{5(x-3)}$ are united, with a subtraction symbol $\boxed{-}$, a common factor can be seen in both terms $3x\boxed{\text{(x-3)}} - 5\boxed{(x-3)}$. This common factor $\boxed{(x-3)}$ is pulled out to arrive at $(x-3)(3x-5)$.

AC Method Step 5 Solve each binomial factor for x

The last step is to set the binomial factors equal to zero to solve for x. So $x-3=0$ can be rewritten as $x-\cancel{3}+\cancel{3}=0+3$ and this leads to $\boxed{x=3}$. Then the second binomial $3x-5=0$ can be rewritten as $3x-\cancel{5}+\cancel{5}=0+5$, which is the same as $3x=5$ then both sides are divided by 3, as in $\frac{\cancel{3}x}{\cancel{3}}=\frac{5}{3}$ so $\boxed{x=\frac{5}{3}}$. The solutions are $x=3$ and $x=\frac{5}{3}$.

Solved Problem 2.31 Solve $8x^2-10x+3=0$ by way of the AC method.

AC Method Step 1: Identify factors of $a\times c$ that add up to b

Table 2.4: AC Method $8x^2-10x+3$ $a=8$ $c=3$ $b=-10$

$ac=3\times 8=24$	$b=-10$	$ac=3\times 8=24$	$b=-10$
1 and 24	$1+24=25$	3 and 8	$3+8=11$
-1 and -24	$-1-24=-25$	-3 and -8	$-3-8=-11$
2 and 12	$2+12=14$	4 and 6	$4+6=10$
-2 and -12	$-2-12=-14$	$\boxed{-4}$ and $\boxed{-6}$	$-4-6=\boxed{-10}$

Only the factors -4 and -6 add up to -10.

AC Method Step 2 Rewrite the quadratic equation with 4 terms

The original quadratic equation is rewritten with four terms, by making use of the factors, -4 and -6, $8x^2-4x-6x+3$. The term $4x$ has more in common with $8x^2$ than 3. The term $6x$ has more in common with 3 than with $8x^2$.

AC Method Step 3 Group the terms and factor the 2 groups

Terms can be grouped in this way with a subtraction symbol $(8x^2-4x)\boxed{-}(6x\boxed{-}3)$. When parentheses are added, the last addition symbol changed to a subtraction symbol! Then this equation is factored.

The group $8x^2-4x$ is the same as $\boxed{4\text{x}}(2x)-\boxed{4\text{x}}(1)$. Pull out the greatest common factor $4x$ to make $\boxed{4x(2x-1)}$.

The group $6x-3$ is the same as $3(2x)-3(1)$. In $\boxed{3}(2x)-\boxed{3}(1)$ pull out the greatest common factor 3 to make $\boxed{3(2x-1)}$

AC Method Step 4 Factor once more

A common factor can be seen in both terms $4x\boxed{\text{(2x-1)}}-3\boxed{(2x-1)}$. This common factor $\boxed{(2x-1)}$ is pulled out to arrive at $(4x-3)(2x-1)$.

AC Method Step 5 Solve each binomial factor for x

The binomial $2x-1=0$ can be rewritten as $2x-\cancel{1}+\cancel{1}=0+1$, which is the same as $2x=1$. Then both sides are divided by 2, as in $\frac{\cancel{2}x}{\cancel{2}}=\frac{1}{2}$ so $\boxed{x=\frac{1}{2}}$. $4x-3=0$ can be rewritten as $4x-3+3=0+3$ and this leads to $4x=3$. Then both sides are divided by 4, as in $\frac{\cancel{4}x}{\cancel{4}}=\frac{3}{4}$ so $\boxed{x=\frac{3}{4}}$. $\boxed{\text{Final Answers are } x=\frac{1}{2} \text{ and } x=\frac{3}{4}}$ ■

This chapter demonstrates the following methods of solving quadratic equations.

- Sign recognition
- AC method
- Factoring formulas

Now, let's explore factoring formulas, which include the "difference of squares" and other quadratic factoring formulas. These formulas can simplify factoring when expressions exhibit specific structures. Here are three quadratic factoring formulas:

Definition 2.11 — Factoring Formulas.

$A^2-B^2=(A-B)(A+B)$ .... Difference of squares
$A^2-2AB+B^2=(A-B)^2$ .... Square of a difference
$A^2+2AB+B^2=(A+B)^2$ .... Square of a sum

The first factoring formula, above, is called the difference of squares factoring formula. In this formula, one can see the variables A and B, which can be any two terms.

If one finds a quadratic equation where two squared quantities are subtracted, the difference in squares can be used. If a quadratic equation fits the left side of one of the three factoring formulas above, then the right side of the factoring formula will provide the factors.

The equation $x^2-9=0$ is a difference of squares, in which $9=3^2$, so the first factoring formula can be used to factor it. Which leads to $(x-3)(x+3)$.

Solved Problem 2.32 Apply **factoring formulas**. Factor the quadratic equation $9x^2-64$

Does this expression look like the left side of any of the factoring formulas?

Recall, problem-solving and analysis begins by thinking in terms of components.

- The expression has 2 terms
- The first term has $3x$ squared, $(3x)^2 = 9x^2$
- The second term has 8 squared, $8^2 = 64$
- There is a subtraction symbol

The expression suggests that the following factoring formula applies.

$$\boxed{A^2 - B^2} = (A-B)(A+B) \qquad \text{Difference of squares}$$

To make sure that this expression matches the left side of the formula above, one needs to confirm that each term is produced by squaring a component

$$9x^2 - 64 = \boxed{(3x)}^2 - \boxed{8}^2 = A^2 - B^2 \qquad \text{A=}3x \qquad \text{B=}8$$

This expression does match the left side of the factoring formula, and A and B have been identified. Now the right side of the formula provides the factors.

$$A^2 - B^2 = \boxed{(A-B)(A+B)}$$

The last step is to just plug A and B into the right side of the formula!

$$(A-B)(A+B) = (\boxed{} - \boxed{})(\boxed{} + \boxed{}) = (3x-8)(3x+8)$$

$$\boxed{\text{Final Answer} \qquad \text{The factors are} \quad (3x-8)(3x+8)}$$

A key observation, in the last solved problem, is that the quadratic equation $9x^2 - 64$ is a binomial, because it has 2 monomials or 2 terms. $\boxed{9x^2} - \boxed{64}$. This leads to binomial factors with addition and subtraction as shown in $(x - \ )(x + \ )$ or $(x + \ )(x - \ )$.

Factoring, is a key step to solving a quadratic equation. In the last solved problem, once the factors are found one can solve for the solution of the quadratic equation. Recall that solutions of quadratic equations are points where the equation is equal to zero. Solutions are points where $f(x)$ or y is equal to zero. If the factors are $(3x-8)(3x+8)$, each binomial is solved for zero.

With the first binomial, first, one adds 8 to both sides, $3x - 8 \boxed{+8} = 0 \boxed{+8}$. Then, one divides both sides by 3, as in, $\dfrac{3x}{\boxed{3}} = \dfrac{8}{\boxed{3}}$, which leads to $\boxed{x = \dfrac{8}{3}}$.

With the second binomial, first, one subtracts 8 from both sides, $3x + 8 \boxed{-8} = 0 \boxed{-8}$. Then, one divides both sides by 3 $\dfrac{3x}{\boxed{3}} = \dfrac{-8}{\boxed{3}}$, which leads to $x = \dfrac{-8}{3} = \boxed{x = -\dfrac{8}{3}}$.

Solved Problem 2.33 **Factoring Formulas** - Factor the quadratic equation

$$9x^2 + 24x + 16 = 0$$

The following factoring formula can be used. Why?

$$\boxed{A^2 + 2AB + B^2} = (A + B)^2 \qquad \text{Square of a sum}$$

One needs to confirm that the square root of the first term A^2 shows up in the second term, $2AB$.

$$9x^2 = 3x \times 3x = 9x^2 \qquad \sqrt{9x^2} = 3x \text{ or } -3x \qquad A = 3x$$

$$16 = 4 \times 4 = 16 \qquad \sqrt{16} = \boxed{4} \text{ or -4} \qquad B = \boxed{4}$$

$$24x = 2 \times 12x = 2\,\boxed{(3x)}\,\boxed{(4)} = 2AB$$

In the factoring formula, above, the first term is A^2 and its square root is A. One can see that A is found in the second term $2AB$. In the factoring formula the third term is B^2, and its square root is B. One can see that B is found in the second term $2AB$. So the expression $9x^2 + 24x + 16 = 0$ can be written in the form of the left side of the factoring formula. Recall that $4 \times 4 = 16^2$

$$A^2 + 2AB + B^2 = \boxed{3x}^2 + 2\,\boxed{3x}\,\boxed{4} + \boxed{4}^2$$

Now the right side of the factoring formula provides the factors. The last step is to plug A and B into the right side of the factoring formula!

$$A = \boxed{\text{3x}} \qquad B = \boxed{4} \qquad (A + B)^2 = (\boxed{} + \boxed{})^2 = (3x + 4)(3x + 4)$$

$$\boxed{\text{Final Answer} \quad \text{The factors are} \quad (3x + 4)^2}$$

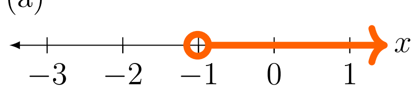
$$3x + 4 - 4 = 0 - 4 \qquad 3x = -4 \qquad \frac{\cancel{3}x}{\cancel{3}} = \frac{-4}{3} \qquad \boxed{x = \frac{-4}{3}}$$

2.6 Inequalities and Quadratics Problems

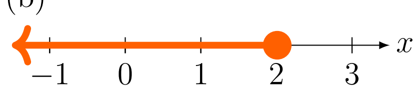
Click on the page number to the right to see the chapter discussion. Click on the solution number to navigate to the solution.

1. Write the inequality for each example below: (Page 67) (Solution 1)

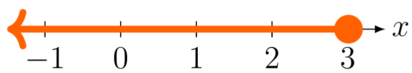
(a)



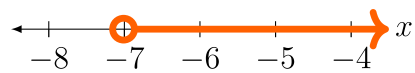
(b)



(c)



(d)



2. Solve the following inequalities. (Page 69) (Solution 2)

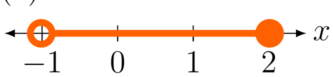
   (a) $-4x > 16$ (b) $\dfrac{x}{-10} \geq 10$ (c) $\dfrac{x}{2} \leq 100$

3. Solve the following inequalities. (Page 68) (Solution 3)

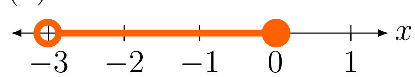
   (a) $x - 78 \geq 26$ (b) $x - 100 < 33$ (c) $x - 15 \leq 45$

4. Write the compound inequality for each example below: (Page 69) (Solution 4)

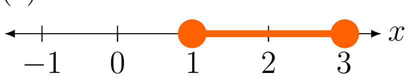
   (a)



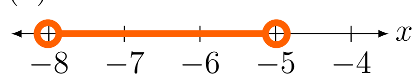
   (b)



   (c)



   (d)



5. Solve the following compound inequalities. (Page 70) (Solution 5)

   (a) $-2 \leq x + 50 \leq 104$ (b) $-40 \leq x - 10 < 50$ (c) $-21 < x + 7 \leq 7$

6. Solve the following compound inequalities. (Page 70) (Solution 6)

   (a) $-25 < \dfrac{x}{-4} < -10$ (b) $-10 < -5x \leq -5$ (c) $-4 \leq \dfrac{x}{-4} \leq 2$

7. Describe the following inequalities with set-builder notation. (Page 71) (Solution 7)

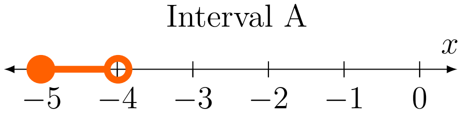
   (a) $-100 \leq x < 5$ (b) $15 < x < 45$ (c) $1000 \leq y$

8. Describe the following inequalities with interval notation.(Page 72) (Solution 8)

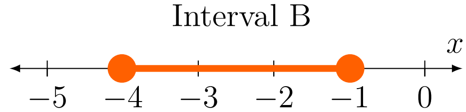
   (a) $12 < x \leq 96$ (b) $-100 \leq x < -95$ (c) $-10 < y$

9. Describe $A \cup B$ with interval notation. (Page 75) (Solution 9)

Interval A

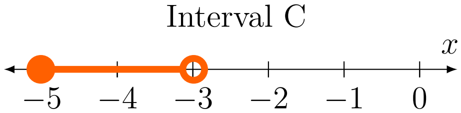


Interval B

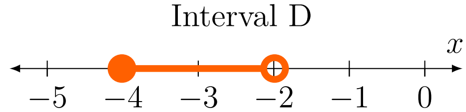


10. Describe $C \cup D$ with interval notation. (Page 75) (Solution 10)

Interval C

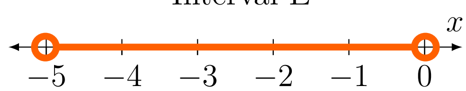


Interval D

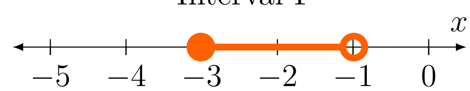


11. Describe $E \cap F$ with interval notation. (Page 75) (Solution 11)

Interval E



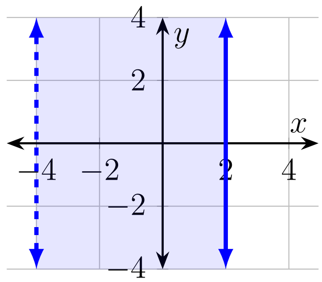
Interval F



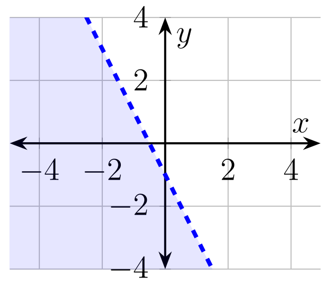
12. Match the graphs below with the following inequalities:

$y < -2x - 1$ $\quad$ $y > 2$ $\quad$ $-4 < x \leq 2$ $\quad$ (Page 77) (Solution 12)

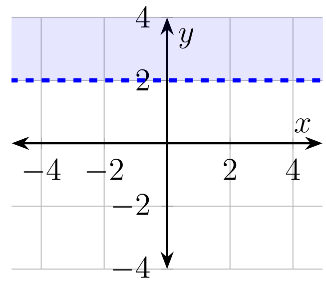
Graph A



Graph B

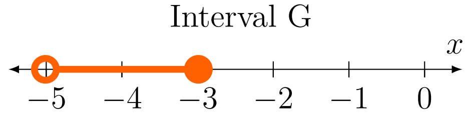


Graph C

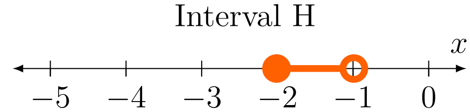


13. Describe $G \cap H$ with interval notation. (Page 75) (Solution 13)

Interval G

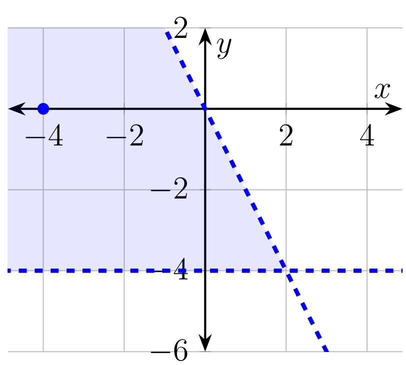


Interval H



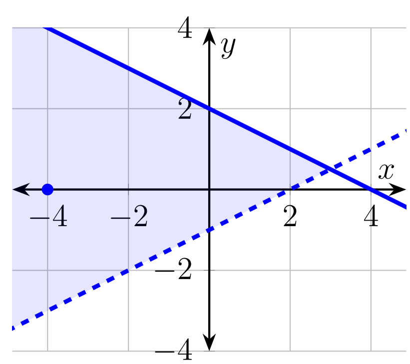
14. Identify the system of inequalities for the graph below.

(Page 79) (Solution 14)



15. Identify the system of inequalities for the graph below.

(Page 79) (Solution 15)

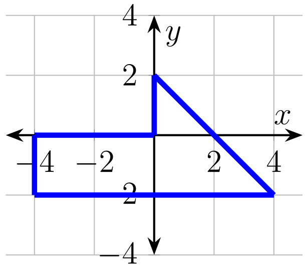


16. Evaluate the following expressions: (Page 67) (Solution 16)

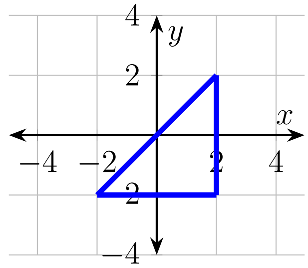
(a) $3^2 \times (2+3) \times (1+2)$ $\quad$ (b) $(5-2) \times 2^3 \times (9-7)$

17. Which graphs below exhibit symmetry? (Page 82) (Solution 17)

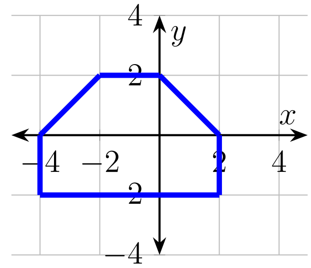
Graph D



Graph E

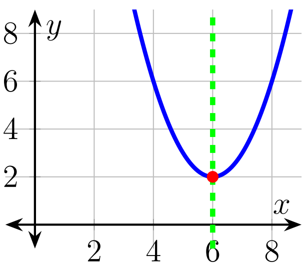


Graph F

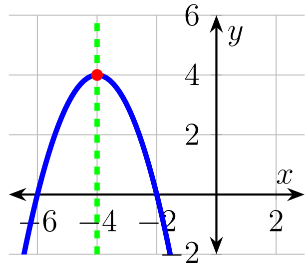


18. For each graph, identify the vertex of the parabola. (Page 83) (Solution 18)

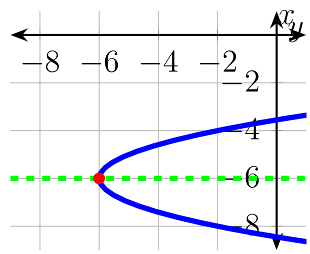
Graph H



Graph I

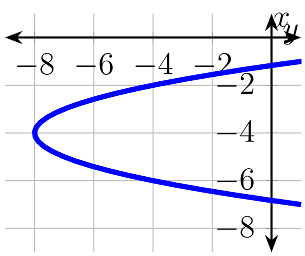


Graph J



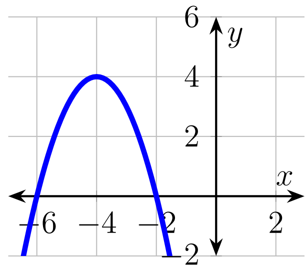
19. For the graph below, describe the domain and range in interval form.

(Page 83) (Solution 19)



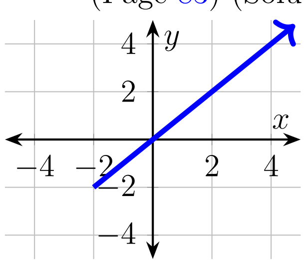
20. For the graph below, describe the domain and range in interval form.

(Page 83) (Solution 20)



21. For the graph below, describe the domain and range in interval form.

(Page 83) (Solution 21)



22. Solve these quadratic equations.

(Page 89) (Solution 22)

(a) $\dfrac{x^2}{5} - 5 = 0$

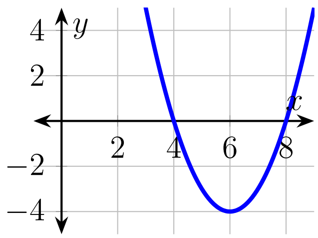
(b) $\dfrac{x^2}{2} - 32 = 0$

23. Multiply the following binomials. (Page 90) (Solution 23)

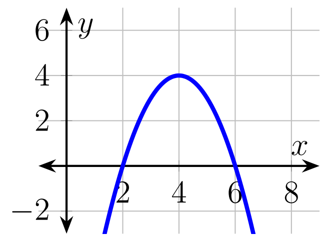
    (a) $(2x+10)(2x+10)$ (b)$(x-6)(x+6)$ (c)$(4x-6)(4x+2)$

24. Find the solutions of the quadratic equations, graphed below. (Page 87) (Solution 24)

    (a)

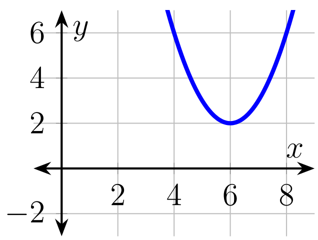


    (b)

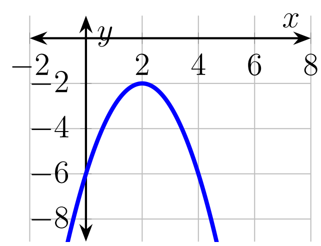


25. Find the solutions of the quadratic equations, graphed below. (Page 87) (Solution 25)

    (a)



    (b)



26. Factor and solve the quadratic equations below.

    (a) x^2-9 (b)x^2-16 (Page 89) (Solution 26)

27. Solve the quadratic equation $x^2-18x+81$. (Page 97) (Solution 27)

28. Solve the quadratic equation x^2-x-20. (Page 97) (Solution 28)

29. Solve the quadratic equation $6x^2+7x+2$. (Page 100) (Solution 29)

30. Solve the quadratic equation $2x^2+7x-15$. (Page 100) (Solution 30)

2.7 Inequalities and Quadratics Solutions

1. (a) $-1 < x$ (b) $x \leq 2$ (c) $x \leq 3$ (d) $-7 < x$

2. (a) $x < -4$ (b) $x \leq -100$ (c) $x \leq 200$

3. (a) $x \geq 104$ (b) $x < 133$ (c) $x \leq 60$

4. (a) $-1 < x \leq 2$ (b) $-3 < x \leq 0$ (c) $1 \leq x \leq 3$ (d) $-8 < x < -5$

5. (a) $-52 \leq x \leq 54$ (b) $-30 \leq x < 60$ (c) $-28 < x \leq 0$

6. (a) $40 < x < 100$ (b) $1 \leq x < 2$ (c) $-8 \leq x \leq 16$

7. (a) $\{x \mid -100 \leq x < 5\}$ (b) $\{x \mid 15 < x < 45\}$ (c) $\{y \mid 1000 \leq y\}$

8. (a) $(12, 96]$ (b) $[-100, -95)$ (c) $(-10, +\infty)$

9. $[-5, -1]$

10. $[-5, -2)$

11. $[-3, -1)$

12. Graph A is $-4 < x \leq 2$ Graph B is $y < -2x - 1$ Graph C is $y > 2$

13. $\emptyset$ or $\{\}$

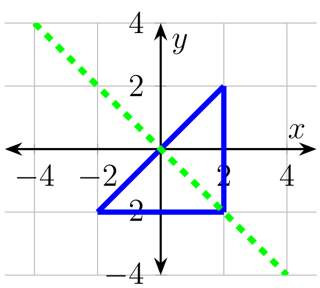
14. $y > -4$ and $y < -2x$

15. $y > \frac{x}{2} - 1$ and $y \leq -\frac{x}{2} + 2$

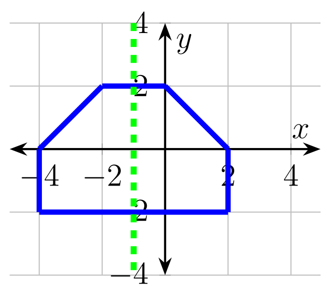
16. (a) 135 (b) 48

17. Graph E and Graph F exhibit symmetry. The axis of symmetry is shown, for each graph.

Graph E



Graph F



18. Graph H (6,2) Graph I $(-4, 4)$ Graph J $(-6, -6)$

19. Domain $[-8, +\infty)$, Range $(-\infty, +\infty)$

20. Domain $(-\infty, +\infty)$, Range $(-\infty, 4]$

21. Domain $[-2, +\infty)$, Range $[-2, +\infty)$

22. (a) $x = \pm 5$ (b) $x = \pm 8$

23. (a) $4x^2 + 40x + 100$ (b) $x^2 - 36$ (c) $16x^2 - 16x - 12$

24. Solutions are $x = 4$ and $x = 8$ (b) Solutions are $x = 2$ and $x = 6$

25. (a) No solution (b) No solution

26. (a) Solutions are $x = 3$ and $x = -3$ (b) Solutions are $x = +4$ and $x = -4$

27. Solution is $x = 9$

28. Solutions are $x = -4$ and $x = 5$

29. Solutions are $x = -\frac{2}{3}$ and $x = -\frac{1}{2}$

30. Solutions are $x = \frac{3}{2}$ and $x = -5$

Chapter 3: Quadratics II and Complex Numbers

OVERVIEW

The sections of this chapter are:

3.1 The Quadratic Formula
3.2 Completing the Square
3.3 Visual Complex Numbers
3.4 Complex Conjugation is Simple!
3.5 Complex Number Operations
3.6 Complex Quadratic Equations

A quadratic equation has been discussed and demonstrated. In the previous chapter the following methods were applied to solve quadratic equations:

- Sign recognition
- AC method
- Factoring formulas

It has been emphasized that solving quadratic equations means finding points where the quadratic equation is equal to zero. Solutions to quadratic equations are also called zeros or roots. It may be necessary to make use of some additional methods to solve quadratic equations. In this chapter, additional methods of solving quadratic equations will be applied to solve quadratic equations:

- Quadratic formula
- Square root property
- Completing the square

It will be eye-opening to take a look at complex numbers in a visual way. **Complex numbers can be useful in powerful ways, but there is an approach to complex numbers that can be quite simple**. Complex numbers help with some interesting and mysterious behavior and phenomena, that shows up every day in the world around us. With complex numbers it is possible to:

- Implement 3-D video games
- Design with inductors and capacitors that are found in almost all electronic devices
- Describe different characteristics of light
- Understand audio amplifiers and speakers
- Describe power and electricity usage in homes and businesses
- Understand medical equipment and sensors

OBJECTIVES

By the end of the chapter, a student will be able to:

- Apply the quadratic formula to solve quadratic equations
- Apply the square root property to solve quadratic equations

- Apply the complete the square method to solve quadratic equations
- Discuss imaginary numbers
- Graph complex numbers
- Solve complex quadratic equations

3.1 The Quadratic Formula

A quadratic equation was discussed and defined in Definition 2.10. Recall that a quadratic equation will have the following form.

$$ax^2 + bx + c = 0$$

One can see, just above, that the quadratic equation is set to zero. When a quadratic equation is set equal to zero in this manner, then the quadratic formula can be used to solve it. The use of the quadratic formula is demonstrated hereafter. First, it is worthwhile to practice placing quadratic equations in the form shown, just above. This can involve some of the methods that were discussed in Subsection 1.3 Solving for an Unknown. Solving for an unknown is demonstrated in Solved Problem 1.9.

Solved Problem 3.1 Modify the following quadratic equations so that they are set equal to zero.

(a) $-x^2 + 6x = -18$ (b) $3x^2 = -4x - 2$ (c) $5x^2 = -6x - 1$

(a) $-x^2 + 6x = -18$ A first step is to eliminate -18 from the right side of the equation. Recall that there is a strong association between addition and subtraction. Addition is used to eliminate -18 from the right side. So, 18 is added to both sides of the equation.

$$-x^2 + 6x + 18 = -18 + 18 \qquad \boxed{-x^2 + 6x + 18 = 0}$$

(b) $3x^2 = -4x - 2$ A first step is to eliminate -2 from the right side of the equation, by adding 2 to both sides. Then it is necessary to eliminate -4x from the right side of the equation, by adding $4x$ to both sides.

First, 2 is added to both sides.

$$3x^2 = -4x - 2 \qquad 3x^2 + 2 = -4x - 2 + 2 \qquad 3x^2 + 2 = -4x$$

Then, $4x$ is added to both sides of the equation.

$$3x^2 + 2 + 4x = -4x + 4x \qquad \boxed{3x^2 + 4x + 2 = 0}$$

(c) $5x^2 = -6x - 1$ First, 1 is added to both sides.

$$5x^2 + 1 = -6x - 1 + 1 \qquad 5x^2 + 1 = -6x$$

Then, $6x$ is added to both sides.

$$5x^2+1+6x=-6x+6x \qquad \boxed{5x^2+1+6x=0}$$

It has been demonstrated that a function, works like a simple system with one input and one output. In other words, a function is like a helpful tool, that given an input, produces an output. The quadratic formula also works like a tool, when solving quadratic equations.

Definition 3.1 — Quadratic Formula. For a quadratic equation in the form $ax^2+bx+c=0$, the quadratic formula provides the solutions.

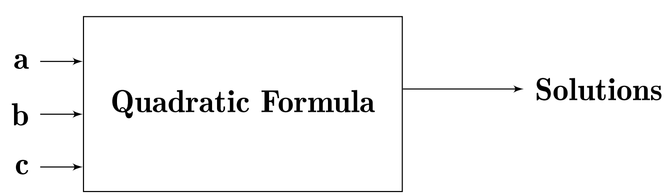
The quadratic formula has the form:

$$\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

The inputs to the quadratic formula are the coefficients a, b, and c. The outputs of the quadratic equation are values of x that are solutions. The **discriminant** is b^2-4ac.

The figure below, emphasizes that the quadratic formula is simply a tool. The inputs to this tool are the coefficients a, b, and c from the quadratic equation. The outputs of this tool will be the solutions of the quadratic equation. From the figure below, one can gather that to use the quadratic formula, a first step would be to identify the coefficients of the quadratic equation. These coefficients are plugged in, and the quadratic formula provides the solutions. Recall that a quadratic equation can have 1 solutions, 2 solutions, or no solutions. Solutions of quadratic equations were visually demonstrated in Solved Problem 2.21

Fig. 3.1: Quadratic Formula Inputs and Outputs



Tip To use the quadratic formula the quadratic equation must be equal to zero

Solved Problem 3.2 Quadratic Formula - Solve $2x^2-10x=-3$

First, this quadratic equation is set equal to zero. So 3 is added to both sides to make the equation equal to zero.

$2x^2-10x\ \boxed{+3} = -3\ \boxed{+3}$ which leads to $2x^2-10x+3=0$

Second, the coefficients a, b, and c are identified.

$\boxed{2}\,x^2 - \boxed{10}\,x + \boxed{3} = 0$ the coefficients are $\boxed{a=2}$, $\boxed{b=-10}$, and $\boxed{c=3}$

The coefficients are plugged into the quadratic equation. $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$x = \frac{-\boxed{\text{b}} \pm \sqrt{\boxed{\text{b}}^2 - 4\,\boxed{\text{a}}\,\boxed{\text{c}}}}{2\,\boxed{\text{a}}} \qquad x = \frac{-\boxed{-10} \pm \sqrt{\boxed{-10}^2 - 4\,\boxed{(2)}\,\boxed{3}}}{2\,\boxed{(2)}}$$

$$-(-10) = 10 \quad (-10)^2 = 100 \qquad x = \frac{10 \pm \sqrt{100-24}}{4} \qquad x = \frac{10 \pm \sqrt{76}}{4}$$

The symbol $\pm$ means "plus or minus". So there are 2 equations to solve.

$$x = \frac{10+\sqrt{76}}{4} \quad \text{and} \quad x = \frac{10-\sqrt{76}}{4} \qquad \sqrt{76} = 8.717$$

$$x = \frac{10+8.717}{4} = \frac{18.717}{4} = \boxed{4.679 = x} \qquad x = \frac{10-8.717}{4} = \frac{1.283}{4} = \boxed{.321 = x}$$

$$\boxed{\text{Final Answers are} \quad x = 4.679 \text{ and } x = .321}$$

These solutions are said to be approximations, in decimal form. There is an alternative route where solutions are exact, and this is discussed below. ▪

In the last solved problem, if $\sqrt{76}$, is plugged into a calculator, the result is not a whole number. The result will be in decimal form. The following examples demonstrate square roots where the square root is not a whole number.

$$\sqrt{66} = \pm 8.12 \qquad \sqrt{38} = \pm 6.16 \qquad \sqrt{11} = \pm 3.32$$

The following examples demonstrate square roots where the square root is a whole number.

$$\sqrt{64} = \pm 8 \qquad \sqrt{36} = \pm 6 \qquad \sqrt{16} = \pm 4$$

Some books or courses will avoid or ignore this point, and it leaves students very confused. There is a practical, useful, take-away point here. When dealing with radical expressions, if the radical expression is not equal to a whole number, there are two simple choices. It is not very difficult to choose.

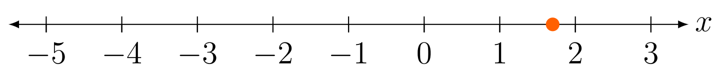
- The solution in radical form (simplified) - exact solution
- The solution in decimal form (calculator) - approximate solution

The expression $\sqrt{38} = \pm\ 6.16$ can be said to have 2 components $\boxed{\sqrt{38}} = \boxed{\pm\ 6.16}$.

Based on the bullet points, just above, one can choose to write the solution as $\sqrt{38}$, or one can choose to write the solution as ± 6.16. The component $\sqrt{38}$ on the left is said to describe an exact value. It is said to be an exact solution. The component on the right ± 6.16, is in decimal form, and it is said to be an approximate solution, or an approximation. Why?

In a quick experiment, if one plugs $\sqrt{38}$ into a calculator it will give a value that starts with 6.164 followed by many digits or decimal places. Notice that in writing/printing 6.164, here, only 3 digits after the decimal are used. The calculator is TRYING to describe a number and doing so requires MANY digits. A calculator will need to use many decimal places to provide accuracy, BUT it is still an approximation. On the other hand $\sqrt{38}$ describes all this in an exact way.

The solutions in the last solved problem are correct, but they are in decimal form, and they are also approximations. Briefly consider the following number line.



In the number line above, one might need to describe the location of the orange dot. One could say that it is between the tick mark 1 and the tick mark 2, or the integer 1 and 2, and this would be correct. If one only uses the integers above, one cannot really describe the location with accuracy. Sure, one can say that it seems to be close to 1.5, or a little greater than 1.5, but this would not be very accurate. One can add 10 divisions in between zero and one. Still, if the orange dot is very small then the orange dot would still be between 2 tick marks. One can keep dividing the space in between tick marks, but the orange dot might still be located between 2 tick marks. In one of the solutions, in the last solved problem, $x = 4.679$ is **much more accurate** than $x = 4.6$, but $x = 4.679$ is still an approximation.

Scientist, engineers, medical researchers, and healthcare workers will very often work with decimals. If more accuracy is needed one simply adds decimal places. In the last solved problem it is easy to plug $\sqrt{76}$ into a calculator to get a decimal. So why discuss this matter? The practical point here is that if one is asked for an "exact solution", decimals should not be used. The solution is written in radical form. **One should simplify the radical form, but it remains as a radical expression**. So, how would the last solved problem be answered if one is asked for exact solutions?

In order to avoid decimals, **keep the radical expression but simplify**. For $\sqrt{76}$ in the last solved problem, one needs to find the highest factor of 76, for which the square root is a whole number. Factors of 76 that have a whole number square root are 1 and 4. The numbers 9 and 16 are not factors of 76. So 4 is the highest factor of 76 that has a whole number square root. This means that $\sqrt{76}$ can be rewritten as $\sqrt{4(19)}$. This can be simplified further.

Recall that $\sqrt{76} = \sqrt{4(19)} = \sqrt{4}\sqrt{19}$, so this can be simplified and rewritten as $\sqrt{76} = \boxed{2\sqrt{19}}$. Notice that $\sqrt{19}$ was not simplified and it was left as is, because its square root is not a whole number. The last solved problem required solving the following 2 equations.

$$x = \frac{10 + \boxed{\sqrt{76}}}{4} \quad \text{and} \quad x = \frac{10 - \boxed{\sqrt{76}}}{4} \qquad \text{now } \sqrt{76} = 2\sqrt{19}$$

These 2 equations are then rewritten using $2\sqrt{19}$

$$x = \frac{10 + \boxed{2\sqrt{19}}}{4} \quad \text{then simplify} \quad \frac{\cancel{2}}{\cancel{2}}\left(\frac{5 + 1\sqrt{19}}{2}\right) = \boxed{\frac{5 + \sqrt{19}}{2}}$$

$$x = \frac{10 - \boxed{2\sqrt{19}}}{4} \quad \text{then simplify} \quad \frac{\cancel{2}}{\cancel{2}}\left(\frac{5 - 1\sqrt{19}}{2}\right) = \boxed{\frac{5 - \sqrt{19}}{2}}$$

Just above, the exact solutions have an orange border. In these exact solutions decimals are not used, they are avoided.

Tip If one is asked for exact answers, the solutions are written in radical form and simplified

Just above, there is a simplification step that is quite simple with practice, but not at first.

Why is $\dfrac{10 + 2\sqrt{19}}{4}$ multiplied by $\dfrac{2}{2}$?

In $\dfrac{\boxed{10} + \boxed{2}\sqrt{19}}{\boxed{4}}$ one can see the numbers 10, 2, and 4. These numbers have a factor in common that is used to simplify them.

For 10, the factors are 1, $\boxed{2}$, 5, and 10 For 2, the factors are 1 and $\boxed{2}$

For 4, the factors are 1, $\boxed{2}$, and 4

So the greatest common factor of 10, 2, and 4, is 2.

$$\frac{10 + 2\sqrt{19}}{4} \text{ is the same as } \frac{\boxed{2}(5) + \boxed{2}(1)\sqrt{19}}{\boxed{2}(2)} \text{ or } \frac{\cancel{2}}{\cancel{2}}\left(\frac{5 + \sqrt{19}}{2}\right)$$

The number 2 shows up as a factor in all the terms, and it can be removed.

Solved Problem 3.3 Quadratic Formula - Solve $2x^2 + 9x = 5$

First, this quadratic equation is set equal to zero. So 5 is subtracted from both sides to make the equation equal to zero.

$2x^2+9X\boxed{-5}=5\ \boxed{-5}$ which leads to $2x^2+9x-5=0$

Second, the coefficients a, b, and c are identified.

$\boxed{2}\ x^2+\boxed{9}\ x-\boxed{5}=0$ the coefficients are $\boxed{a=2}$, $\boxed{b=9}$, and $\boxed{c=-5}$

The coefficients are plugged into the quadratic equation. $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x=\frac{-\boxed{\text{b}}\pm\sqrt{\boxed{\text{b}}^2-4\boxed{\text{a}}\boxed{\text{c}}}}{2\boxed{\text{a}}} \qquad x=\frac{-\boxed{9}\pm\sqrt{\boxed{9}^2-4\boxed{(2)}\boxed{-5}}}{2\boxed{(2)}}$$

$$9^2=81 \qquad x=\frac{-9\pm\sqrt{81-(-40)}}{4} \qquad x=\frac{-9\pm\sqrt{121}}{4} \qquad 81-(-40)=121$$

The symbol $\pm$ means "plus or minus". So there are 2 equations to solve.

$$x=\frac{-9+\sqrt{121}}{4} \quad \text{and} \quad x=\frac{-9-\sqrt{121}}{4} \qquad \sqrt{121}=11$$

Notice that the $\sqrt{121}$ **is equal to a whole number**. It is not necessary to choose between an exact answer and a decimal answer.

$$x=\frac{-9+11}{4}=\frac{2}{4}=\boxed{\frac{1}{2}=x} \qquad x=\frac{-9-11}{4}=\frac{-20}{4}=\boxed{-5=x}$$

$$\boxed{\text{Final Answers are}\quad x=\frac{1}{2}\ \text{and}\ x=-5}$$

Solved Problem 3.4 Quadratic Formula - Solve $x^2+4x=3$

First, this quadratic equation is set equal to zero. So 3 is subtracted from both sides to make the equation equal to zero.

$x^2+4x\ \boxed{-3}=3\ \boxed{-3}$ which leads to $x^2+4x-3=0$

Second, the coefficients a, b, and c are identified.

$\boxed{1}\ x^2+\boxed{4}\ x\ \boxed{-3}=0$ the coefficients are $\boxed{a=1}$, $\boxed{b=4}$, and $\boxed{c=-3}$

The coefficients are plugged into the quadratic equation. $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x = \frac{-\boxed{b} \pm \sqrt{\boxed{b}^2 - 4\boxed{a}\boxed{c}}}{2\boxed{a}} \qquad x = \frac{-\boxed{4} \pm \sqrt{\boxed{4}^2 - 4\boxed{(1)}\boxed{-3}}}{2\boxed{(1)}}$$

$$4^2 = 16 \qquad x = \frac{-4 \pm \sqrt{16-(-12)}}{2} \qquad x = \frac{-4 \pm \sqrt{28}}{2}$$

Notice that $\sqrt{28}$ is **not equal to a whole number**. So, it is necessary to choose between an exact answer and a decimal answer. The symbol $\pm$ means "plus or minus". So, there are 2 equations to solve.

$$x = \frac{-4 \boxed{+} \sqrt{28}}{2} \quad \text{and} \quad x = \frac{-4 \boxed{-} \sqrt{28}}{2} \qquad \sqrt{28} = \sqrt{4}\sqrt{7} = \boxed{2\sqrt{7}}$$

$$x = \frac{-4 \boxed{+} 2\sqrt{7}}{2} = \frac{-2(2)+2\sqrt{7}}{2(1)} = \frac{2}{2}\left(\frac{-2+1\sqrt{7}}{1} = 2+\sqrt{7}\right) = \boxed{-2+\sqrt{7}}$$

$$x = \frac{-4 \boxed{-} 2\sqrt{7}}{2} = \frac{-2(2)-2\sqrt{7}}{2(1)} = \frac{2}{2}\left(\frac{-2-1\sqrt{7}}{1} = 2-\sqrt{7}\right) = \boxed{-2-\sqrt{7}}$$

$$\boxed{\text{Final Answers are} \quad -2+\sqrt{7} \text{ and } -2-\sqrt{7}}$$

Solved Problem 2.20 and Solved Problem 2.21 demonstrated the solutions or roots of a quadratic equation, in a visual way. It is possible to know in advance about the solutions of a quadratic equation. It was mentioned in Definition 3.1 that a, b, and c are coefficients of the quadratic equation, and that $\boxed{b^2 - 4ac}$ is also called the discriminant. This discriminant can be negative, zero, or positive, and this can provide information about the solutions of a quadratic equation.

For example if $x^2 - 3x + 4 = 0$ then $\boxed{b^2 - 4ac} = 9 - 16 = -7$, which is negative

For example if $x^2 - 2x + 1 = 0$ then $\boxed{b^2 - 4ac} = 4 - 4 = 0$, which is zero

For example if $x^2 + 5x + 2 = 0$ then $\boxed{b^2 - 4ac} = 25 - 8 = 17$, which is positive

Tip The discriminant $b^2 - 4ac$ can be used to know how many solutions there are for a quadratic equation

An axis of symmetry was defined in Definition 2.6, and it was demonstrated in Solved Problem 2.16. The vertex of a parabola was defined in Definition 2.7, and it was demonstrated in Solved Problem 2.17. The coefficients of a quadratic equation can also be used to solve for the vertex and axis of symmetry, for a quadratic equation.

Table 3.1: Discriminant $b^2 - 4ac$

Negative $b^2 - 4ac < 0$	Zero $b^2 - 4ac = 0$	Positive $b^2 - 4ac > 0$
2 complex roots	1 real root	2 real roots
No real solutions	1 x-intercept	2 x-intercepts
No real solutions	1 real solution	2 real solutions

Definition 3.2 — Axis of Symmetry Formula. For a quadratic function, $ax^2 + bx + c = y$, with the coefficients a, b, and c. The axis of symmetry is defined by $\boxed{x = \frac{-b}{2a}}$.

Definition 3.3 — Vertex Formula. For a quadratic function, $ax^2 + bx + c = y$, with the coefficients a, b, and c. The x-coordinate of the vertex is defined by $\boxed{x = \frac{-b}{2a}}$. This last equation is used to solve for x, then the quadratic function is used to solve for the y-coordinate.

Solved Problem 3.5 Solve for the axis of symmetry and the vertex of the quadratic function $x^2 - 4x + 3 = y$.

For the quadratic function $\boxed{1}\,x^2\,\boxed{\text{-4}}\,x + \boxed{3} = y$, $a = 1$, $b = -4$, and $c = 3$.

Quadratic functions were discussed on page 86. The coefficients are plugged into the axis of symmetry formula.

$$x = \frac{-b}{2a} = \frac{-(-4)}{2(1)} = \frac{4}{2} = \boxed{2 = x}$$

The equation $x = 2$ describes a line that is the axis of symmetry. Now the input, output behavior of a quadratic function can be used find the value of y at $x = 2$. This x value is then plugged into the initial quadratic equation.

$$\boxed{\text{x}}^2 - 4\,\boxed{\text{x}} + 3 = y \qquad 2^2 - 4(2) + 3 = 4 - 8 + 3 = y \qquad \boxed{-1 = y}$$

Final Answers: The axis of symmetry is $x = 2$. The vertex is $(2, -1)$

3.2 Completing the Square

Recognizing the square and square root of a quantity can make it easier to solve a quadratic equation. A quadratic equation has the form $ax^2+bx+c=0$. Now, if the coefficient a is 1, and b is equal to zero, and if c is negative, then the quadratic equation takes the form $ax^2-c=0$, which can be rewritten in the form $x^2=c$. A square root can then be applied to solve for the solutions.

Solved Problem 3.6 Square Root Property - Solve $x^2-4=0$

The equation $x^2-4=0$ can be rewritten as $x^2=4$

To solve it, one finds the square root of both sides. Recall that $\sqrt{x^2}=\pm x$ and $\sqrt{4}=\pm 2$

$\sqrt{x^2}=\sqrt{4}$ is rewritten as $x=\pm 2$ $\boxed{\text{Final Answers } x=\pm 2}$ ■

In the last solved problem, solving the quadratic equation involved isolating the term x^2, and finding the square root. A similar approach can be used to solve a full quadratic equation. In the method called completing a square, one is creating a quantity where a quantity is squared, and a square root can be applied. This is learned best by applying it. The following steps help while learning to apply the completing the square method.

Step 1 Start with the quadratic equation set equal to zero, and $a=1$
Step 2 Isolate the variable terms
Step 3 Solve the formula $\left(\frac{b}{2}\right)^2$
Step 4 Add this last value to both sides of the equation
Step 5 Factor the left-hand side
Step 6 Solve for the variable x

Solved Problem 3.7 Completing the Square - Solve $x^2+4x+1=0$

Completing the square step 1: Start with the quadratic equation set equal to zero, and $a=1$

The equation $x^2+4x+1=0$ is already equal to zero. The a coefficient is also already equal to 1, so this step is complete.

Completing the square step 2: Isolate the variable terms

This requires subtracting 1 from both sides, as in $x^2+4x+1-1=0-1$ to arrive at $x^2+4x+\boxed{}=-1$.

Completing the square step 3: Solve the formula $\left(\frac{b}{2}\right)^2$

The coefficient b is equal to 4. This coefficient is plugged into the equation $\left(\frac{\boxed{}}{2}\right)^2$ which leads to $\left(\frac{4}{2}\right)^2 = (2)^2 = \boxed{4}$

Completing the square step 4: Add this last value to both sides of the equation

$x^2+4x+\boxed{4} = -1+\boxed{4}$ leads to $\boxed{x^2+4x+4=3}$

Completing the square step 5: Factor the left-hand side

At this point the left side will factor easily, and both factors will be the same. Since $2\times 2 = 4$ and $2+2 = 4$, factors of 4 that add up to 4 are 2 and 2. So the factors of the left-hand side will be $(x+2)(x+2)$.

Completing the square step 5: Solve for the variable x

The equation is then rewritten, $(x+2)^2 = 3$. Finding the square root of both sides leads to $\sqrt{(x+2)^2} = \sqrt{3}$, which is rewritten as $x+2 = \pm 3$. The variable x is isolated by subtracting 2 from both sides $x+2-2 = \pm\sqrt{3}-2$, which leads to $x = \pm\sqrt{3}-2$.

$\boxed{\text{Final Answers} \qquad x = \sqrt{3}-2 \text{ and } x = -\sqrt{3}-2}$ ■

3.3 Visual Complex Numbers

At this point, it is worthwhile to demonstrate complex numbers in a visual way. Complex has a particular meaning in everyday language, but the term complex in math and algebra has a different meaning. In math, algebra, and science the term complex is related to a sum of components. This use of the term complex is very different from the common, everyday use of the term complex in plain English. In everyday, common, Plain English complex is associated with complicated or difficult. In math and algebra, complex should be associated with simply a sum of simpler pieces or components! This is discussed and confirmed further hereafter.

These first chapters have emphasized the value of thinking in terms of components. Complex numbers are the combination of real numbers and imaginary numbers, and they have the form shown below.

Definition 3.4 — Complex Number. A complex number is a sum of a real number and an imaginary number.

A complex number has the form:

$Z = a + ib \qquad Z = \boxed{\text{a}} + \boxed{\text{bi}}$

The term $\boxed{a}$ is the real term. The term $\boxed{bi}$ is the imaginary term.

Complex numbers can build on previous understanding. Consider that a coordinate pair has 2 components, the x-coordinate, and the y-coordinate. Complex numbers can be graphed, in a similar way. The complex plane is shown, below, to the right. One can see that it is a lot like the regular coordinate plane. Again, there is a horizontal axis, and a vertical axis. The horizontal axis is now called the real axis, and it is labeled as "Re". The vertical axis is now called the imaginary axis, and it is labeled as "Im". In some ways, the complex plane is the same concept as the coordinate plane. The complex plane is also called the Argand plane to honor a French mathematician.

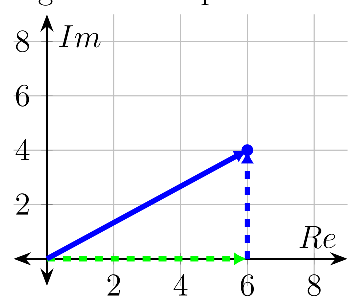
$6 + 4i \qquad \boxed{6} + \boxed{4i}$

$\boxed{6}$ is graphed like an x-coordinate

$\boxed{4i}$ is graphed like a y-coordinate

Graphing $6 + 4i$ is quite similar to graphing the coordinate pair (6, 4)

Fig. 3.2: Complex Plane



In the graph, just above, one can see that the complex number is graphed as the blue dot. The blue dot has 2 components. There is a real, horizontal component of 6 units that is shown with the green, dashed arrow. There is an imaginary, vertical component of 4 units that is shown with the blue, dashed line. Notice the solid, blue arrow. The solid blue arrow is called a vector. The solid, blue vector has a length, that is also called its magnitude. This blue arrow, also, has a clear direction. **The complex number is simply the blue dot, but a complex number is often represented and treated as a vector**. Vectors are used a lot in different ways, but at this time it is key to simply remember that a vector looks like an arrow.

Definition 3.5 — Vector. A vector has a **direction** and a **magnitude**. It is represented by a directed line segment, that looks just like an arrow. One way to describe a vector is with the notation (x, y), much like a coordinate pair. The concept of a vector is visualized as a 2-dimensional line, much like an arrow.

A complex number is a lot like a coordinate pair, which is not difficult at all. So why are complex numbers termed this way? The term "complex" is often used to describe a combination of 2 or more components. The term "complex" as a combination of simpler pieces shows up over and over. It is not necessary to discuss sound or light waves in detail, but a combination of 2 or more simple light waves is called a complex wave. This doesn't make the wave "difficult" it is just a combination of more than one simple wave. It is not necessary to discuss B vitamins in detail, but a combination of several B vitamins is called B complex. This doesn't make B complex "difficult" it is simply a combination of simpler B vitamins. A complex number combines a real component and an imaginary component, and due to this combination it is called a complex number.

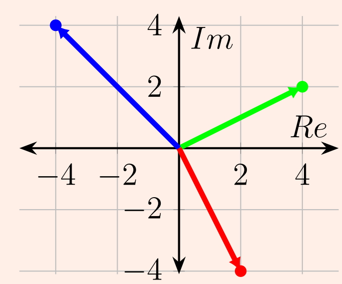
Solved Problem 3.8 Graph the complex numbers $4+2i$, $-4+4i$, $2-4i$

$4+2i$ From the origin go 4 to the right and 2 up, shown with a green dot

$-4+4i$ From the origin go 4 to the left and 4 up, shown with a blue dot

$2-4i$ From the origin go 2 to the right and 4 down, shown with a red dot

Final Answer



Each complex number can be represented by a dot or a vector. ■

One can see that complex numbers are not difficult to work with, and they can be a lot like working with a coordinate pair. A student may see that, but the student might still wonder about the imaginary part.

Almost everyone has experienced or played with a magnet. There are different kinds of magnets, but this brief discussion will keep the focus on algebra and complex numbers. Magnets have several characteristics and one of these characteristics makes use of complex numbers. In everyday terms one can say that a magnet has magnetism. One can not directly see this magnetism, but this magnetism is very real. This magnetism can be measured, and indirectly observed. Powerful magnets can injure fingers if mishandled. When a magnet is described the imaginary part of a complex number is very useful. The term "imaginary" in imaginary number is not the best term, and in fact in other fields it is replaced with other terms like "reactive", because very real behavior and phenomena are being described. **A takeaway point is that complex numbers are like a tool, and like any tool one just needs to learn how to use it**.

3.4 Complex Conjugation is Simple!

Complex in the context of complex numbers has been explained, and it has nothing to do with difficulty. Likewise, complex conjugation is simple. Realizing that anything in algebra can be explained in plain English, should push a student forward through these chapters with the motivation that anything in algebra can be understood well.

It is clear that a complex number has the form $a+bi$. For an example of a complex number such as $2-5i$, the complex conjugate simply changes the sign. This means that the real part of the complex number stays the same and the imaginary part of the complex number will have the opposite sign. The complex conjugate of $-2-5i$ would be $-2+5i$. The complex conjugate of $7+33i$ would be $7-33i$.

Solved Problem 3.9 Find the complex conjugate of these complex numbers:
(a) $10+50i$ (b) $5-19i$ (c) $-15+95i$ (d) $-55-39i$

For each complex number the imaginary part is the second term. For each complex number simply change the sign of ONLY the imaginary part, to the opposite.

Final Answers: (a) $10-50i$ (b) $5+19i$ (c) $-15-95i$ (d) $-55+39i$

3.5 Complex Number Operations

This section will demonstrate addition, subtraction, multiplication, and division of complex numbers. Multiplication and division of complex numbers will make use of complex conjugation, that was demonstrated in the last section.

In complex number addition and subtraction, the real and imaginary parts of a complex number are handled separately. For example for complex number addition the real parts or components are added separately. Then the imaginary components are added separately. To add $5+10i$ to $4+4i$, the real numbers 5 and 4 are added to make 9. Then the imaginary numbers $10i$ and $4i$ are added to make $14i$. Thus, the sum or final result is $9+14i$.

Solved Problem 3.10 Solve the expressions

(a) $(10+50i)+(13+30i)$ (b) $(4+12i)+(44+4i)$

(a) In $\boxed{10}+50i+\boxed{13}+30i$ first combine the real components, $10+13=23$. Then, in $10\ +50i\ +\ 13\ +30i$ combine the imaginary components, $50i+30i=80i$.

Final Answer (a) $23+80i$

(b) First, combine the real components, $4+44=48$. Then, add the imaginary components, $12i+4i=16i$. Final Answer (b) $48+16i$

Complex number subtraction is similar to complex number addition, that was demonstrated above. To solve $(100+100i)-(30+30i)$, the subtraction or negative sign is distributed across the second complex number, as shown below. This leads to $100+100i-30-(+30i)$. Recall that $-1\times 30=-30$.

$$(-1)(30+30i) = -30-30i$$

The initial complex numbers are then rewritten, $100+100i-30-30i$. Real and imaginary components are then combined. Real numbers are combined $100-30=70$. Then imaginary numbers are combined $100i-30i=70i$. The result is $\boxed{70+70i}$.

This last example was simple enough, but signs can cause errors. This is worth emphasizing. The 2 complex numbers below are subtracted. **The first subtraction or negative symbol below must be distributed across the second complex number**, as shown below.

$$(80+80i)\boxed{-}(40-40i) \qquad (-1)(40-40i) = -40+40i$$

Distribution of the subtraction symbol completes the subtraction. Then real and imaginary components are combined. This is demonstrated below. Recall that $-1\times -40 = 40$.

$$(80+80i)-(40\ \boxed{-40i}\,) \text{ is rewritten as } 80+80i-40\ \boxed{-(-40i)}$$

This can be rewritten as $80+\boxed{80i}-40\ \boxed{+40i} = \boxed{40+120i}$

Solved Problem 3.11 Solve the expressions

(a) $(20+44i)-(13-30i)$ (b) $(90+80i)-(44-10i)$

(a) Distribute the subtraction, negative sign across the second complex number to write $20+44i-13+30i$. First combine the real components, $20-13=7$. Then combine the imaginary components, $44i+30i=74i$. $\boxed{\text{Final Answer (a) } 7+74i}$

(b) Distribute the subtraction, negative sign across the second complex number to write $90+80i-44+10i$. First combine the real components, $90-44=46$. Then combine the imaginary components, $80i+10i=90i$. $\boxed{\text{Final Answer (a) } 46+90i}$ ■

Multiplying complex numbers builds on the concept of multiplying binomials. Multiplying binomials is based on the distributive property. This is demonstrated below.

$$(2+5i)(2+5i) = 4+10i+10i+25i^2 = 4+20i+25(-1) = 4-25+20i = \boxed{-21+20i}$$

Applying exponents to i leads to either i, $-i$, 1, or -1. Applying exponents to the imaginary number i may lead to a solution that is real(without i) or complex(with i). If $i=\sqrt{-1}$ then:

$$i^2 = \sqrt{-1}\sqrt{-1} = -1 \qquad i^5 = 1(i) = i$$
$$i^3 = -1(i) = -i \qquad i^6 = i^2 i^4 = -1(1) = -1$$
$$i^4 = i^2 i^2 = (-1)(-1) = 1 \qquad i^7 = i^6(i) = -1(i) = -i$$

Solved Problem 3.12 Solve the expressions (a) $(2-i)(2-i)$ (b) $(4-3i)(2-3i)$

(a) $(2-i)(2-i) = 4\ \boxed{-2i-2i} + \boxed{i^2} = 4-4i-1 = \boxed{3-4i}$

Note that $-2i-2i = -4i$ also $i^2 = -1$

(b) $(4-3i)(2-3i) = 8\ \boxed{-12i-6i} + \boxed{9i^2} = 8-18i-9 = \boxed{-1-18i}$

Note that $-12i-6i = -18i$ also $9i^2 = 9(-1) = -9$ ■

The complex conjugate of $2+5i$ is $2-5i$. For the complex number $2+5i$ the complex conjugate can be used to remove the imaginary number $5i$, as demonstrated below. This is done by multiplying the complex number by its complex conjugate. This also involves multiplying binomials that was demonstrated in Solved Problem 2.24.

$$(2+5i)(2-5i) = 4\ \boxed{-10i+10i} + \boxed{(5i)(-5i)} = 4-25i^2 = 4+25 = \boxed{29}$$

The complex conjugate of $2+5i$ is $2-5i$.

Recall that $-10+10i = 0$ also $i^2 = i \times i = \sqrt{-1}\sqrt{-1} = -1$

For this reason $(5i)(-5i) = 5 \times -5 \times i \times i = -25i^2 = -25(-1) = 25$

One can see, just above that a complex conjugate can be used to remove the imaginary number in a complex number. The number 29 is a real number, and it is no longer a complex number, because there is no imaginary number.

$$(2-i)(2+i) = 4\ \boxed{+2i-2i} - \boxed{i^2} = 4-(-1) = \boxed{5}$$

Tip A complex conjugate can be used to remove the imaginary number in a complex number!

Division of complex numbers will make use of a complex conjugate, in order to eliminate an imaginary term. For example, in order to divide $3-6i$ by $1-5i$ it is written as a fraction, then it is multiplied by 1. The number 1 is written in the form of a fraction, where both the numerator and denominator are equal. Both, the numerator and denominator are made equal to the complex conjugate, of the denominator, of the first fraction.

$$\frac{3-6i}{1-5i}=\left(\frac{3-6i}{1-5i}\right)1=\left(\frac{3-6i}{1-5i}\right)\boxed{\frac{(1+5i)}{(1+5i)}}\quad \text{The complex conjugate of } 1-5i \text{ is } 1+5i$$

Next, the distributive property is used to multiply binomials.

For the numerator $(3-6i)(1+5i)=3+15i-6i-30i^2=3+9i-(30)(-1)=\boxed{33+9i}$

For the denominator $(1-5i)(1+5i)=1+5i-5i-25i^2=1-(25)(-1)=1+25=\boxed{26}$

This leads to $\dfrac{33+9i}{26}$ which can be rewritten as $\boxed{\dfrac{33}{26}+\dfrac{9}{26}i}$

Solved Problem 3.13 Evaluate $2+5i$ divided by $4-i$

Begin with $\dfrac{2+5i}{4-i}$ the complex conjugate of $4-i$ is $4+i$. Then, multiply both the numerator and the denominator by this complex conjugate. $\left(\dfrac{2+5i}{4-i}\right)\boxed{\dfrac{(4+i)}{(4+i)}}$

The numerators $(2+5i)(4+i)=8+2i+20i+5i^2=8+22i+(5)(-1)=\boxed{3+22i}$

The denominators $(4-i)(4+i)=16+4i-4i-i^2=16-(-1)=17=\boxed{17}$

This leads to $\dfrac{3+22i}{17}$ rewritten as $\boxed{\text{Final Answer}\quad \dfrac{3}{17}+\dfrac{22}{17}i}$ ■

3.6 Complex Quadratic Equations

Table 3.1 emphasized that the discriminant $\boxed{b^2-4ac}$ may be negative, zero, or positive. This can be used to know whether there are no real solutions, 1 solution, or 2 solutions for the quadratic equation. When the discriminant is negative there are complex solutions. Both the quadratic formula and the completing the square method can be used to solve complex quadratic equations. Complex solutions simply include the imaginary number i. For the case where there are complex solutions, and no real solutions the quadratic function does not touch the x-axis, so it does not have an x-intercept.

Solved Problem 3.14 Quadratic Formula - Solve $x^2 - 3x + 10 = 0$

First, this quadratic equation is set equal to zero. This equation is already set equal to zero. Second, the coefficients a, b, and c are identified.

$\boxed{1}\, x^2 \boxed{-3}\, x \boxed{+10} = 0$ the coefficients are $\boxed{a = 1}$, $\boxed{b = -3}$, and $\boxed{c = 10}$

The coefficients are plugged into the quadratic equation... $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-\boxed{\text{b}} \pm \sqrt{\boxed{\text{b}}^2 - 4\boxed{\text{a}}\boxed{\text{c}}}}{2\boxed{\text{a}}} \qquad x = \frac{-\boxed{-3} \pm \sqrt{\boxed{-3}^2 - 4\boxed{(1)}\boxed{10}}}{2\boxed{(1)}}$$

$$-(-3) = 3 \quad (-3)^2 = 9 \qquad x = \frac{3 \pm \sqrt{9 - 40}}{2} \qquad x = \frac{3 \pm \sqrt{-31}}{2}$$

The symbol $\pm$ means "plus or minus". So there are 2 equations to solve...

$$x = \frac{3 + \sqrt{-31}}{2} \quad \text{and} \quad x = \frac{3 - \sqrt{-31}}{2} \qquad \sqrt{-31} = \sqrt{31}\sqrt{-1} = \sqrt{31}i$$

$x = \dfrac{3 + \sqrt{-31}}{2}$ can be rewritten as $x = \dfrac{3}{2} + \dfrac{\sqrt{31}}{2}i$

$x = \dfrac{3 - \sqrt{-31}}{2}$ can be rewritten as $x = \dfrac{3}{2} - \dfrac{\sqrt{31}}{2}i$

Final Answers: $x = \dfrac{3}{2} + \dfrac{\sqrt{31}}{2}i$ and $x = \dfrac{3}{2} - \dfrac{\sqrt{31}}{2}i$

It has been explained that a complex number is just a combination of a real term and an imaginary term. In the last solved problem, in one of the solutions...

$x = \boxed{\dfrac{3}{2}} + \boxed{\dfrac{\sqrt{31}}{2}i}$ the real component is $\dfrac{3}{2}$ and the imaginary component is $+\dfrac{\sqrt{31}}{2}i$.

In the last solved problem, two solutions are shown. These two solutions can be rewritten in the following way, which would still describe two solutions ... $x = \frac{3}{2} \pm \frac{\sqrt{31}}{2}i$. Recall that the symbol $\pm$ means "plus or minus", so this last expression actually describes two solutions.

Solved Problem 3.15 Quadratic Formula - Solve $x^2 + 2x + 3 = 0$

First, this quadratic equation is set equal to zero. This equation is already set equal to zero. Second, the coefficients a, b, and c are identified.

$\boxed{1}\, x^2 \boxed{+2}\, x \boxed{+3} = 0$ the coefficients are $\boxed{a = 1}$, $\boxed{b = 2}$, and $\boxed{c = 3}$

The coefficients are plugged into the quadratic equation... $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-\boxed{\text{b}} \pm \sqrt{\boxed{\text{b}}^2 - 4\boxed{\text{a}}\boxed{\text{c}}}}{2\boxed{\text{a}}} \qquad x = \frac{-\boxed{2} \pm \sqrt{\boxed{2}^2 - 4\boxed{(1)}\boxed{3}}}{2\boxed{(1)}}$$

$$x = \frac{-2 \pm \sqrt{4 - 12}}{2} \qquad x = \frac{-2 \pm \sqrt{-8}}{2} \qquad \sqrt{-8} = \sqrt{4}\sqrt{2}\sqrt{-1} = 2\sqrt{2}i$$

$$x = \frac{-\boxed{2} \pm \boxed{2}\sqrt{2}i}{\boxed{2}} \quad \text{is simplified to } x = \frac{-1 \pm 1\sqrt{2}i}{1} \qquad \boxed{\text{Final Answers: } -1 \pm i\sqrt{2}}$$

This last solved problem, made use of a simplification step...

$$x = \frac{-\boxed{2} \pm \boxed{2}\sqrt{2}i}{\boxed{2}} \qquad x = \frac{2(-1 \pm 1\sqrt{2}i)}{2(1)} = \frac{2}{2}\left(\frac{-1 \pm 1\sqrt{2}i}{1}\right) = \frac{-1 \pm 1\sqrt{2}i}{1} = -1 \pm \sqrt{2}i$$

Sure, this simplification step is quite simple with practice, but a student might find that practice is needed with this step. If this simplification step is confusing, it can make the whole problem SEEM difficult. Spending a little time with this simplification step will help throughout an algebra course.

In the numerator $-\boxed{2} \pm \boxed{2}\sqrt{2}i$, 2 is found in both terms, this can be pulled out, so the numerator becomes $2(-1 \pm \sqrt{2})$.

In the denominator, 2 is also a factor, and 2 is equal to $\boxed{2}(1)$.

The expression $\frac{2}{2}x = x$ could be rewritten as $\boxed{\frac{3}{3}x = \frac{\cancel{3}}{\cancel{3}}x = x}$ or $\boxed{\frac{5}{5}x = \frac{\cancel{5}}{\cancel{5}}x = x}$

The fractions $\frac{2}{2}$, $\frac{3}{3}$, and $\frac{5}{5}$ are equal to 1. Since $\frac{2}{2} = 1$, the expression can be simplified...

$$\frac{\cancel{2}}{\cancel{2}}\left(\frac{-1\pm 1\sqrt{2}i}{1}\right)=\frac{-1\pm 1\sqrt{2}i}{1}=-1\pm\sqrt{2}i$$

The quadratic formula has been applied to complex quadratic equations. Complex quadratic equations can also be solved by completing the square.

Solved Problem 3.16 Completing the Square - Solve $3x^2-12x+17=0$

Completing the square step 1: Start with the quadratic equation set equal to zero, and $a=1$

The equation $3x^2-12x+17=0$ is already equal to zero. The a coefficient is not equal to 1, so divide all by 3.

$\frac{3x^2}{3}-\frac{12x}{3}+\frac{17}{3}=\frac{0}{3}$ simplifies to $x^2-4x+\frac{17}{3}=0$

Completing the square step 2: Isolate the variable terms

This requires subtracting $\frac{17}{3}$ from both sides, as in $x^2-4x+\frac{17}{3}\boxed{-\frac{17}{3}}=0\boxed{-\frac{17}{3}}$ to arrive at $x^2-4x+\boxed{}=-\frac{17}{3}$.

Completing the square step 3: Solve the formula $\left(\frac{b}{2}\right)^2$

The coefficient b is equal to -4. This coefficient is plugged into the equation $\left(\frac{\boxed{}}{2}\right)^2$ which leads to $\left(\frac{-4}{2}\right)^2=(-2)^2=\boxed{4}$

Completing the square step 4: Add this last value to both sides of the equation

$x^2-4x+\boxed{4}=-\frac{17}{3}+\boxed{4}$ leads to $\boxed{x^2-4x+4=-\frac{17}{3}+\frac{12}{3}}$ $\quad -\frac{17}{3}+\frac{12}{3}=-\frac{5}{3}$

Completing the square step 5: Factor the left-hand side

At this point the left side will factor easily, and both factors will be the same. Since $-2\times-2=4$ and $-2-2=-4$, factors of 4 that add up to -4 are -2 and -2. So the factors of the left-hand side will be $(x-2)(x-2)$.

Completing the square step 5: Solve for the variable x

The equation is then rewritten, $(x-2)^2 = -\frac{5}{3}$. Finding the square root of both sides leads to $\sqrt{(x-2)^2} = \pm\sqrt{\frac{-5}{3}}$, which is rewritten as $x-2 = \pm\sqrt{\frac{5}{3}}i$. The variable x is isolated by adding 2 to both sides $x-2+2 = \pm\sqrt{\frac{5}{3}}i+2$, which leads to $x = 2\pm\sqrt{\frac{5}{3}}i$.

Multiplying anything by 1 does not change it, and $1 = \frac{\sqrt{3}}{\sqrt{3}}$

$\sqrt{\frac{5}{3}} \times 1 = \frac{\sqrt{5}}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{15}}{3}$

$\boxed{\text{Final Answers} \qquad x = 2 \pm \frac{\sqrt{5}}{3}i}$ ■

In this last solved problem, the form of the solution makes it clear that the solution has a real term and an imaginary term. In this last solved problem, there are 2 solutions.

One solution is $x = 2+\frac{\sqrt{15}}{3}i$, and the other solution is $x = 2-\frac{\sqrt{15}}{3}i$.

In $x = 2-\frac{\sqrt{15}}{3}i$, the real term is 2, and the imaginary term is $dfrac\sqrt{15}3i$.

In $x = 2-\frac{\sqrt{15}}{3}i$, the real term is 2, and the imaginary term is $-dfrac\sqrt{15}3i$.

Since $2 = \frac{6}{3}$, the solution can also be written as $x = \frac{6}{3} \pm \frac{\sqrt{15}}{3}i$ or $x = \frac{6\pm\sqrt{15}i}{3}$.

3.7 Quadratics II and Complex Numbers Problems

Click on the page number to the right to see the chapter discussion. Click on the solution number to navigate to the solution.

Solve the following quadratic equations.

1. (a) $x^2-9=0$ (b) $x^2-25=0$ (c) $x^2-169=0$
   (Page 118) (Solution 1)

2. (a) $x^2-9=7$ (b) $x^2-30=6$ (c) $x^2-56=8$
   (Page 118) (Solution 2)

3. How many real solutions are there for the equation $x^2+5x+2=0$?
   (Page 116) (Solution 3)

4. How many real solutions are there for the equation $x^2 - 3x + 4 = 0$?
(Page 116) (Solution 4)

Solve the following quadratic equations.

5. $x^2 - 5x - 14 = 0$
(Page 111) (Solution 5)

6. $3x^2 - x - 2 = 0$
(Page 111) (Solution 6)

7. $27x^2 - 12 = 0$ (Page 114) (Solution 7)

8. $x^2 - x = 6$ (Page 114) (Solution 8)

9. Solve for the exact solutions of $2x^2 + 12x - 1 = 0$ (Page 115) (Solution 9)

10. Solve for the exact solutions of $-x^2 - 8x + 2 = 0$ (Page 115) (Solution 10)

11. Find the axis of symmetry and vertex of $x^2 + 6x + 5 = 0$. (Page 117) (Solution 11)

12. Find the axis of symmetry and vertex of $2x^2 - 4x + 5 = 0$. (Page 117) (Solution 12)

Solve the following quadratic equations.

13. $x^2 - 3x - 18 = 0$
(Page 118) (Solution 13)

14. $x^2 + 8x + 2 = 22$
(Page 118) (Solution 14)

15. $x^2 - 4x - 12 = 0$
(Page 118) (Solution 15)

16. Graph the following complex numbers, as a dot or a vector.
(a) $2 + 6i$ (b) $8 + 2i$ (c) $4 + 6i$ (Page 121) (Solution 16)

17. Graph the following complex numbers, as a dot or a vector.
(a) $6 - 4i$ (b) $2 - 4i$ (c) $8 - 6i$ (Page 121) (Solution 17)

18. Find the complex conjugate of the following complex numbers.
(a) $5 + 70i$ (b) $72 - 3i$ (Page 122) (Solution 18)

19. Solve the following expressions.
(a) $(25 + 22i) + (20 + 10i)$ (b) $(15 + 17i) + (12 - 7i)$ (Page 122) (Solution 19)

20. Solve the following expressions.
(a) $(200 + 22i) - (100 + 11i)$ (b) $(76 + 88i) - (26 - 12i)$ (Page 123) (Solution 20)

21. Solve the following expressions.
(a) $2(2 + 6i)$ (b) $10(10 + 10i)$ (Page 124) (Solution 21)

22. Solve the following expressions.
(a) $(3 + 6i)(5 + 6i)$ (b) $(9 + 7i)(8 + 7i)$ (Page 124) (Solution 22)

23. Evaluate the following expressions.
    (a) $\frac{2-i}{3i}$ (b) $\frac{-2}{1+i}$ (Page 125) (Solution 23)

24. Evaluate the following expressions.
    (a) $\frac{2+5i}{4-i}$ (b) $\frac{3+4i}{8-2i}$ (Page 125) (Solution 24)

25. Solve the quadratic equation $x^2+2x+5=0$. (Page 126) (Solution 25)

26. Solve the quadratic equation $x^2+4x+5=0$. (Page 126) (Solution 26)

27. Solve the quadratic equation $x^2-4x+13=0$. (Page 126) (Solution 27)

28. Solve the quadratic equation $x^2+4x+7=0$. (Page 128) (Solution 28)

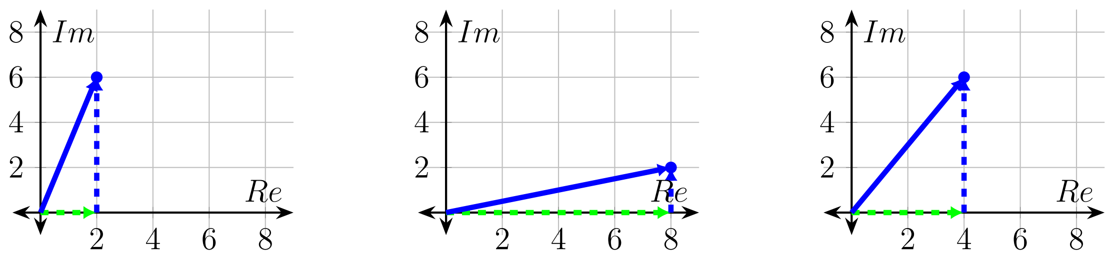
29. Solve the quadratic equation $9x^2-12x+9=0$. (Page 128) (Solution 29)

30. Solve the quadratic equation $2x^2-6x+20=0$ (Page 128) (Solution 30)

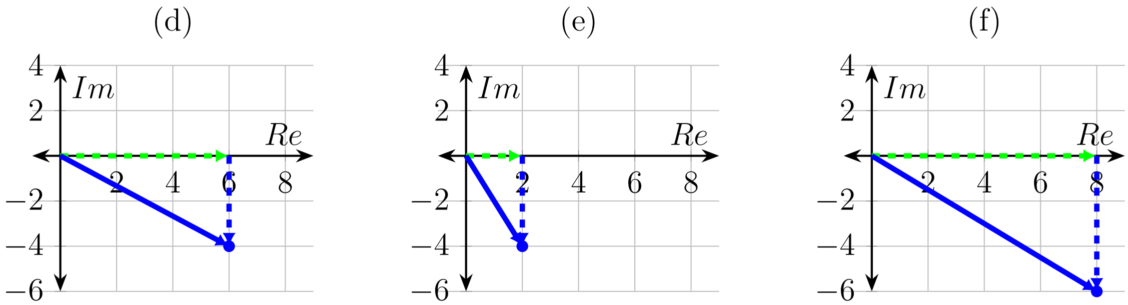
3.8 Quadratics II and Complex Numbers Solutions

1. (a) $x=\pm 3$ (b) $x=\pm 5$ (c) $x=\pm 13$
2. (a) $x=\pm 4$ (b) $x=\pm 6$ (c) $x=\pm 8$
3. $b^2-4ac=17$, which is positive, 2 real solutions
4. $b^2-4ac=-7$, which is negative, no real solutions
5. $x=-2$ and $x=7$
6. $x=1$ and $x=-\frac{2}{3}$
7. $x=\pm\frac{2}{3}$
8. $x=3$ and $x=-2$
9. $x=\frac{-6\pm\sqrt{38}}{2}$
10. $x=-4\pm 3\sqrt{2}$
11. Axis of symmetry is $x=-3$. The vertex is (-3, -4)
12. Axis of symmetry is $x=1$. The vertex is (1, 3)
13. $x=6$ and $x=-3$
14. $x=2$ and $x=-10$
15. $x=6$ and $x=-2$
16.

(a) (b) (c)



17.



18. (a) $5-70i$ (b) $72+3i$

19. (a) $45+32i$ (b) $27+10i$

20. (a) $100+11i$ (b) $50+100i$

21. (a) $4+12i$ (b) $100+100i$

22. (a) $-21+48i$ (b) $23+119i$

23. (a) $-\frac{1}{3}-\frac{2}{3}i$ (b) $-1+i$

24. (a) $\frac{3}{17}+\frac{22}{17}i$ (b) $\frac{8}{34}+\frac{19}{34}i$

25. $-1\pm 2i$

26. $-2\pm i$

27. $2\pm i3$ or $2\pm 3i$

28. $-2\pm i\sqrt{3}$ or $-2\pm\sqrt{3}i$

29. $\frac{2}{3}\pm i\frac{\sqrt{5}}{3}$ or $\frac{2}{3}\pm\frac{\sqrt{5}}{3}i$

30. $\frac{3}{2}\pm i\frac{\sqrt{31}}{2}$ or $\frac{3}{2}\pm\frac{\sqrt{31}}{2}i$

Chapter 4: Higher Degree Polynomials

OVERVIEW

The sections of this chapter are:

4.1 Applied Exponents
4.2 Polynomials
4.3 Adding and Subtracting Polynomials
4.4 Multiplying Polynomials
4.5 Dividing Polynomials
4.6 Graphing Higher Order Polynomials

Linear equations and quadratic equations were discussed previously. A key difference between a linear equation and a quadratic equation is that a quadratic equation made use of an exponent of 2. Linear functions and straight lines are simple, limited cases. Blueprints or plans for structures and living spaces do make use of a lot of straight lines. Now try to imagine a car, a boat, or an airplane with only straight lines. It can be said, that, most of what is interesting in finance, science, and nature, displays curves and nonlinear behavior. Both curves and nonlinear behavior have everything to do with polynomials.

OBJECTIVES

By the end of the chapter, a student will be able to:

- Understand simple interest in basic banking
- Understand the role of exponents in compound interest
- Describe and communicate with polynomials
- Add and subtract polynomials
- Multiply and divide polynomials
- Visually interpret and analyze polynomials

4.1 Applied Exponents

It is clear that an exponent of 2 is one of the key characteristics of a quadratic function. It has been discussed and demonstrated that a quadratic function is nonlinear. An exponent of 2 makes it possible to describe a curve, and more specifically an exponent of 2 makes it possible to describe a parabola.

Tip Exponents make it possible to describe a variety of curved lines

Beyond a quadratic equation, and beyond an exponent of 2, exponents make it possible to describe a variety of curves and nonlinear behavior. When working with exponents, the fundamental order of operations often plays a role.

Curved lines and nonlinear behavior show up everywhere. Recall that nonlinear refers to curved lines, as opposed to straight lines. One application of exponents, curved lines, and nonlinear behavior, is interest, as it relates to savings and lending.

Definition 4.1 — Interest. Interest is the concept of a reward or compensation for lending money, for the lender. For the borrower, it is the cost of borrowing money.

When one deposits money at a bank or financial institution this is basically lending money to the bank. On the other hand, one might borrow money from the bank, when purchasing a vehicle, a home, or land. There can be many rules when one lends money or borrows money, but here the focus will be on simple interest and compound interest.

Definition 4.2 — Simple Interest. This would be the cost of a loan if one is borrowing. If one is lending this would be the return on the loan. It is abbreviated as SI, below.

$$SI = PRT$$

P is principal, or the amount that is borrowed. T is the time span of the loan, **in years**. R is the rate that describes the cost of borrowing, in decimal form.

It should be clarified that in this definition, R for rate can be described in two ways. In lending, a rate can be described as a percentage or in decimal form. **In this definition, the rate must be described in decimal form**.

Solved Problem 4.1 Rewrite these rates in percentage form.

(a) .25 (b) .1 (c) .3 (d) .15

(a) $.25 \times 100 = \boxed{25\%}$

(b) $.1 \times 100 = \boxed{10\%}$

(c) $.3 \times 100 = \boxed{30\%}$

(d) $.15 \times 100 = \boxed{15\%}$ ■

Solved Problem 4.2 Rewrite these rates in decimal form.

(a) 22% (b) 12% (c) 90% (d) 8%

(a) $\frac{22}{100} = \boxed{.22}$ (b) $\frac{12}{100} = \boxed{.12}$ (c) $\frac{90}{100} = \boxed{.9}$ (d) $\frac{8}{100} = \boxed{.08}$ ■

One might see a different formula for simple interest and this is shown below.

$$SI = \frac{PRT}{100} \qquad R \text{ is in percentage form}$$

Both equations represent the same concept but differ in rate format: one in percentage form, the other in decimal form. It's crucial to use the correct rate format in simple interest calculations. This course will use the form shown in Definition 4.2, and this form requires the rate to be in decimal form.

Adam lends 1,000 United States Dollars (USD) to Brian at a rate of 10% for 1 year. The amount borrowed by Brian is 1,000 USD and this is called the principal. The time span that has been agreed upon is, clearly 1 year. Then the cost of the loan is stated as a rate of 10%. The simple interest equation that will be used here requires the rate to be in decimal form. Then the rate of 10% is converted to .1 in decimal form. The calculation of simple interest is shown below.

$$SI = PRT = 1000(.1)1 = \boxed{100 \text{ USD}}$$

An interpretation of this is that Brian borrowed a principal amount of 1000 USD. Brian has to pay back this loan of 1000 USD and will also pay back an additional cost of 100 USD for the loan. Notice that this simple interest equation only provides the cost of the loan for Brian, or the return on the loan for Adam. The result this equation provides does not include the total amount paid or returned.

Solved Problem 4.3 Mary lends 15,000 USD to Jennifer at a rate of 15% for 2 years. What is the simple interest?

First, the rate is converted to decimal form $\frac{15}{100} = .15$. Then simple interest is calculated.

$SI = PRT = 15,000(.15)2 = 4,500$ USD $\boxed{\text{Final Answer 4,500 USD}}$ ■

The simple interest formula doesn't involve exponents, unlike compound interest, which does. For instance, if Mary lends $15,000$ USD and earns $4,500$ USD in simple interest, she has a total of $19,500$ USD. By reinvesting this $19,500$ USD, she earns interest not only on the initial $15,000$ USD but also on the accumulated interest of $4,500$ USD. This leads to a higher return each time she reinvests, illustrating compound interest.

Yes, compounding means that Mary will earn interest on interest, and the growth of Mary's money will be nonlinear. Still, it is simple, intuitive, and very true to realize that **compound interest simply describes a repeated lending process**.

The lending process could repeat every 2 years or every year, every month, or with any time period. The equation for compound interest makes it possible to describe how often this lending process will repeat.

Definition 4.3 — Compound Interest. This describes how a value or currency amount accumulates or grows over time if interest is added periodically, or in a repeated way.

$$A = P\left(1+\frac{R}{n}\right)^{nT}$$

A is the final amount. P is principal, or the amount that is borrowed. T is the time span of the loan, **in years**. R is the rate that describes the cost of borrowing, in **decimal form**. Then n is the number of times that compounding will occur **per year**.

A key observation about the definition of compound interest is that it provides the total of the original amount or principal, plus all the interest gained. The definition of simple interest only provided the interest. Another key observation is that in the definition for compound interest, the interest rate must be converted to decimal form.

In the compound interest formula, n denotes the compounding frequency per year. For instance, if compounding happens once a year, $n=1$, twice yearly is $n=2$, quarterly is $n=4$, and every month $n=12$.

If an initial amount of 10,000 USD receives an interest of 10% per year for 10 years, then the principal P would be 10,000 USD. The rate R is 10%, but it is converted to decimal form, so the rate R is .10, and then used in the equation. The time span, T, in years is 10. Then the lending process repeats or compounds once per year, so n would be 1.

$$A = P\left(1+\frac{R}{n}\right)^{nT} = 10,000\left(1+\frac{.1}{1}\right)^{1(10)} = 25,937.42 \text{ USD}$$

Solved Problem 4.4 For a principal amount of 10,000 USD, interest is paid at 20% every six months, for 10 years. Find the final amount.

First, the rate percentage is converted to decimal form $\frac{20}{100} = .2$. Then n is 2 because compounding occurs every 6 months, which is twice per year.

$$A = P\left(1+\frac{R}{n}\right)^{nT} = 10,000\left(1+\frac{.2}{2}\right)^{2(10)} = 67,275 \text{ USD} \quad \boxed{\text{Final Answer } 67{,}275 \text{ USD}}$$

4.2 Polynomials

A binomial is also called a polynomial. A polynomial is a simple extension of monomials and binomials. Again, thinking in terms of components helps with language, algebra, or any learning endeavor. A component of the word polynomial is "poly". The component "poly" comes from the Greek language, and it literally means "many". So, polynomial means many monomials. Two or more monomials makes a polynomial, so a binomial is also a polynomial.

Definition 4.4 — Polynomial. A polynomial is an algebraic expression with two or more terms, or two or more monomials. Each term will have the form $\boxed{cx^n}$ where c is a coefficient, n is an exponent, and x is the variable.

Examples of polynomials are:
2 terms: $12x-15$
3 terms: x^2-5x+9
4 terms: x^4+x^2+5x+3

Each term may have a variable. Each variable may have an exponent. Any exponent on x will be a non-negative, whole number. Division can not be performed by a variable.

In a polynomial, the **leading term** is the term with the highest exponent on a variable. The **leading coefficient** is the coefficient that is multiplied by the variable with the highest exponent. It is common and customary to list the terms of a polynomial in descending order, based on the exponent on the variable x. A polynomial is defined here first because it is the superset, broader, or more general case. As mentioned above, technically a binomial is also a polynomial. Another example of a polynomial is a trinomial. Likewise, a trinomial consists of three terms or monomials.

Definition 4.5 — Trinomial. A trinomial is a simple algebraic expression with only three terms. A trinomial consists of three monomials.

Each term or monomial may have a variable, and each variable may have an exponent. For each variable, the exponent will be a non-negative, whole number. Division can not be performed by a variable.

A trinomial will have the form:

$$\boxed{c_1x^k+c_2x^j+c_3x^i}$$

c_1, c_2, and c_3 are coefficients, x is a variable, k, j, and i are exponents

Examples of trinomials are:
$x^2-4x+4 \qquad x^2-9x+9$

The definition stated that each term or monomial will have the form $\boxed{cx^n}$, but in any term the exponent n could be zero. If a term has a zero exponent then one will not see a variable in the term. For example, in the polynomial $x^2-5x+\boxed{9}$, the lone coefficient 9 is equal to the term $9x^0$. This is because $9x^0=9(1)=9$.

Solved Problem 4.5 Are the following expressions polynomials?

(a) $5x$ (b) $9x^3+1$ (c) $5x^{(-3)}+3x^2+11$ (d) $\dfrac{1}{x+2}$

(a) $5x$ $\boxed{\text{No, only a monomial}}$

(b) $9x^3+1$ $\boxed{\text{Yes}}$

(c) $5x^{(-3)} + 3x^2 + 11$

$\boxed{\text{No, } x \text{ can not have a negative exponent}}$

(d) $\dfrac{1}{x+2}$

$\boxed{\text{No, division can not be performed by } x}$

Table 4.1: Monomials - Polynomials

Category	Examples
Monomial	$12x$ $\quad 7x^2$ $\quad x$
Binomial	$5x+3y$ $\quad x+13$ $\quad x^2-11x$
Trinomial	x^2-4x+4 $\quad x^2-9x+9$

Factoring formulas where shown on page 101, while discussing how to solve quadratic equations. Additional factoring formulas can be added that are useful with cubic equations.

Definition 4.6 — Factoring Formulas.

$A^2-B^2=(A-B)(A+B)$ .... Difference of squares

$A^2-2AB+B^2=(A-B)^2$ .... Square of a difference

$A^2+2AB+B^2=(A+B)^2$ .... Square of a sum

$A^3-B^3=(A+B)(A^2-AB+B^2)$ ... Difference of cubes

$A^3+B^3=(A-B)(A^2+AB+B^2)$ ... Sum of cubes

Definition 1.2 defines degree. A quadratic equation is defined by its degree of 2, and it has a degree of 2 because the highest exponent for any of its variables is 2. The following examples reinforce that the degree of a polynomial is the highest exponent, of any of its variables.

x^2-5x+9 ... Degree is 2

x^4-7x+2 ... Degree is 4

x^5-5x^3+2x ... Degree is 5

x^2 ... Degree is 2

A polynomial can also be written in the form $f(x)=x^2-4x+4$, where $f(x)$ means "function of x". For example $f(6)$ refers to the corresponding output if the input is 6. This brings to light that a polynomial can have an input and an output. Any value of x can be plugged into the equation to find the corresponding output.

Fig. 4.1: $f(x) = x^2 - 4x - 4$

$$\textbf{If } x = 6 \longrightarrow \boxed{f(x) = x^2 - 4x - 4} \xrightarrow{y = 8}$$

Figure 4.1, above reinforces the input, output nature of a polynomial. For the polynomial $f(x) = x^2 - 4x - 4$, if $x = 6$ then 6 is plugged into the equation $f(6) = \boxed{}^2 - 4\boxed{} - 4$. The equation becomes $f(x) = 6^2 - 4(6) - 4 = 36 - 24 - 4 = 8$.

Solved Problem 4.6 For the polynomial function $f(x) = 3x^2 - 9x + 7$ find $f(3)$ and $f(-2)$

To solve $f(3)$ one plugs 3 into the equation $3\boxed{}^2 - 9\boxed{} + 7$ to rewrite it as

$$3(3)^2 - 9(3) + 7 = 3(9) - 27 + 7 = \boxed{7}$$

For $f(-2)$ plug -2 into the equation $3\boxed{}^2 - 9\boxed{} + 7$ to rewrite it as

$3(-2)^2 - 9(-2) + 7 = 3(4) + 18 + 7 = \boxed{37}$ $\boxed{\text{Final Answers } f(3) = 7 \text{ and } f(-2) = 37}$ ■

4.3 Adding and Subtracting Polynomials

When adding polynomials there is a vertical approach, that may be helpful at first. The example, just below, shows the addition of $5x^2 + 7x + 3$ plus $3x^2 + 2x - 1$. This vertical configuration shown below makes it possible to focus on one term at a time.

Step 1	Step 2	Step 3
$\boxed{5x^2} + 7x + 3$	$5x^2 + \boxed{7x} + 3$	$5x^3 + 7x + \boxed{3}$
$\boxed{3x^2} + 2x - 1$	$3x^2 + \boxed{2x} - 1$	$3x^2 + 2x \boxed{-1}$
$\boxed{8x^2}$	$8x^2 + \boxed{9x}$	$\boxed{8x^2 + 9x + 2}$

In a polynomial, terms that have the same exponent are said to be like terms. Finding and combining like terms helps with adding and subtracting polynomials. In the polynomial $x^2 + \boxed{2x} + \boxed{2x} + \boxed{2x} + 9$, notice that the term $2x$ shows up 3 times. In the term $2x$ the exponent is 1 because $2x = 2x^1$. Each time that the term $2x$ shows up it has an exponent or degree of 1. For this reason, the 3 cases of the term $2x$ can be combined and replaced by $6x$. This is because $2x + 2x + 2x = 6x$. This makes it possible to simplify the expression, and rewrite it as $x^2 + 6x + 9$

Solved Problem 4.7 Simplify the polynomial $x^9+5x^4+3x^4+9x^2-6x^2+2$

The objective here is to simply, find, and combine like terms.

$x^9+\boxed{5x^4}+\boxed{3x^4}+9x^2-6x^2+2$ Note that two of these terms have the same exponent. These terms are combined $5x^4+3x^4=\boxed{8x^4}$, which leads to $x^9+8x^4+9x^2-6x^2+2$

$x^9+8x^4+\boxed{9x^2}-\boxed{6x^2}+2$ Two other terms also have the same exponent. These terms are combined $9x^2-6x^2=\boxed{3x^2}$ $\boxed{\text{Final Answer } x^9+8x^4+3x^2+2}$ ■

When adding and subtracting polynomials, terms with different exponents are handled separately. As was demonstrated above, terms with the same exponent on the variable x can be combined to simplify the equation. Consider the addition of the following polynomials, below.

$$x^2+2x+10 \qquad\qquad 3x^2+5x+12$$

One can see right away, there are terms where the exponent on the variable x is 2. For both polynomials these terms are also called the leading terms. One can also see terms where the exponent on the variable x is 1. As shown below, adding the two polynomials begins by adding the like terms. While adding x^2 and $3x^2$ one is only adding the coefficients 1 and 3.

$$\boxed{x^2}+2x+10+\boxed{3x^2}+5x+12=\boxed{4x^2}+2x+10+5x+12$$

The equation has been rewritten after adding like terms. One can now see that the terms $2x$ and $5x$ are also like terms. Combining like terms can be done a second time as shown below.

$$4x^2+\boxed{2x}+10+\boxed{5x}+12=4x^2+\boxed{7x}+10+12$$

In the equation, shown just above, the constants 10 and 12 can also be combined, to arrive at the result $4x^2+7x+22$. One can see that adding polynomials is really about correctly finding and combining like terms. When adding polynomials, this approach of combining like terms is sometimes referred to as a horizontal approach, but the key point is that one is combining like terms.

Solved Problem 4.8 Add the polynomials $4x^3+7x$ and x^2+10.

Each polynomial has two terms and all 4 terms have different exponents on the variable. For this reason each term stays the same, and these 4 terms are not considered like terms. $\boxed{\text{Final Answer } \ 4x^3+x^2+7x+10}$ ■

The terms do not have to be in order, but it is customary to list the terms in order, from left to right. This is shown in the final answer in the solved problem just above. The exponent on the variable of each term is used to place the terms in order. The term with the highest coefficient on the variable x is placed first. Then, the second term would be the term with the second-highest exponent on x. For example, in the term $7x$ the exponent is 1, so this term was placed third in the final answer.

Solved Problem 4.9 Add the polynomials $x^3 + 4x^2$ and $-9x^2 + 4x + 10$.

Notice that the second polynomial does not have a term with an exponent of 3 on the variable x. The term x^3 is alone, and it is simply listed in the final answer as is. Notice that both polynomials have a term with x^2, so these terms are added.

$$x^3 + \boxed{4x^2} + \boxed{-9x^2} + 4x + 10 = x^3 \boxed{-5x^2} + 4x + 10$$

The equation $x^3 - 5x^2 + 4x + 10$ can no longer be simplified, since all the terms have a different exponent on the x variable. The terms x^3, $4x$, and 10 are simply listed in the final answer. $\boxed{\text{Final Answer } x^3 - 5x^2 + 4x + 10}$ ■

Subtraction of polynomials builds on the same concepts that were demonstrated above. During subtraction of polynomials one needs to distribute the subtraction or negative sign across the second polynomial. Consider the subtraction of the following polynomials below.

$$1 \quad 2 \quad 3$$

$$(x^2 - 4) - (2x^2 + 4x - 6) = ? \qquad -1(2x^2 + 4x - 6) = -2x^2 - 4x + 6$$

This makes it possible to rewrite the expression, as shown below. Now, one can see that two terms with x^2 can be combined. Then, the constants 4 and 6 can be combined.

$$\boxed{x^2} - 4 \boxed{-2x^2} - 4x + 6 = -x^2 - 4x - \boxed{4} - \boxed{6} = -x^2 - 4x - 10$$

Solved Problem 4.10 Solve $(5x^2 - 2x + 1) - (-3x^2 - x - 3)$.

Distribution is used to distribute subtraction or negative across the second polynomial.

$$1 \quad 2 \quad 3$$

$$-1(-3x^2 - x - 3) = 3x^2 + x + 3 \qquad (-1)(-3) = 3 \; (-1)(-1) = 1$$

The two polynomials are then combined and added. Two terms with x^2 are added. Then the terms $2x$ and x are added. Then the constants 1 + 3 are added.

$$\boxed{5x^2} - 2x + 1 + \boxed{3x^2} + x + 3 = 8x^2 - \boxed{2x} + \boxed{x} + 1 + 3$$

$\boxed{\text{Final Answer } 8x^2 - x + 4}$ ■

4.4 Multiplying Polynomials

Multiplying polynomials is an exercise in applying the distributive property.

Solved Problem 4.11 Multiply the following polynomials $x^2(3x^2 - 9x + 7)$.

Recall that $(x^2)(x^2) = x^{2+2} = x^4$ also $x^2(x) = x^{2+1} = x^3$

This is an application of the distributive property.

$$x^2(3x^2 - 9x + 7) = 3x^{2+2} + 9x^{2+1} + 7x^2 \qquad \boxed{\text{Final Answer } 3x^4 + 9x^3 + 7x^2}$$ ■

Solved Problem 4.12 Multiply the following polynomials $(x+3)(2x^2 - 5x + 7)$.

The first polynomial is distributed across the second polynomial. The first term, x, of the first polynomial is distributed across the second polynomial. Then the second term, 3, of the first polynomial is distributed across the second polynomial. Recall that $x(x^2) = x^{1+2} = x^3$

$$(x+3)(2x^2 - 5x + 7) = x(2x^2 - 5x + 7) + 3(2x^2 - 5x + 7)$$

$$= x(2x^2) - x(5x) + 7x + 3(2x^2) - 3(5x) + 3(7)$$

$$= 2x^{1+2} - 5x^2 + 7x + 6x^2 - 15x + 21 = 2x^3 - \boxed{5x^2} + 7x + \boxed{6x^2} - 15x + 21$$

One can see, above, that two terms with x^2 can be combined. Then two more terms with x can be combined, as shown below.

$$2x^3 + x^2 \boxed{+7x}\boxed{-15x} + 21 \qquad \boxed{\text{Final Answer } 2x^3 + x^2 - 8x + 21}$$ ■

4.5 Dividing Polynomials

Dividing polynomials applies addition, subtraction, and multiplication of polynomials. The following concepts will be demonstrated here.

- Long division
- Synthetic division
- Remainder Theorem

Dividing polynomials also applies division of monomials. Division of monomials involves understanding of monomials, and being able to work with exponents. Recall that monomials consist of only one term in the form cx^m. If division of monomials is clear, then dividing polynomials becomes a matter of applying clear steps. Division of monomials is reinforced below.

$$\frac{x^3}{x} = x^3x^{(-1)} = x^{3-1} = x^2 \qquad \frac{-3x^2}{x} = -3(x^2x^{(-1)}) = -3x^{2-1} = -3x$$

Solved Problem 4.13 Evaluate the following expressions:

(a) $\frac{x^4}{x^2}$ (b) $\frac{x^5}{x}$

(a) $\frac{x^4}{x^2} = x^4x^{(-2)} = x^{4-2} = \boxed{x^2}$ (b) $\frac{x^5}{x} = x^5x^{(-1)} = x^{5-1} = \boxed{x^4}$

The best way to learn long division of polynomials is to apply it. The two expressions below, call for division of polynomials.

$$\left(x^3+2x^2-25x-50\right) \div (x+5) \qquad \text{or} \qquad \frac{x^3+2x^2-25x-50}{x+5}$$

The following steps will be helpful while learning long division.

Step 1 Divide the 1^{st} term of the dividend by the 1^{st} term of the divisor
Step 2 The term from Step 1 is multiplied by the divisor
Step 3 The result from Step 2 is subtracted
Step 4 Bring down the next term from the previous dividend
Step 5 Repeat

Long division of polynomials is demonstrated, below. The dividend is $\boxed{x^3+2x^2-25x-50}$. The divisor is $\boxed{x+5}$.

$$\begin{array}{r} x^2 \\ \boxed{x}+5\overline{\big)\boxed{x^3}+2x^2-25x-50} \end{array}$$

Step 1: The 1^{st} term of the dividend $\boxed{x^3}$ is divided by x, the 1^{st} term of the divisor, to arrive at x^2.

$$\boxed{x^2}(x+5) = x^3+5x^2$$

Step 2: The term $\boxed{x^2}$, from Step 1 is multiplied by the divisor.

$$\begin{array}{r} x^2 \\ x+5 \overline{\big) x^3+2x^2-25x-50} \\ \underline{\boxed{-x^3 - 5x^2}} \\ -3x^2 \end{array}$$

Step 3: The result from Step 2 is subtracted, so it becomes $-x^3-5x^2$, as shown.

$$\begin{array}{r} x^2 \\ x+5 \overline{\big) x^3+2x^2-25x-50} \\ \underline{-x^3 - 5x^2} \\ -3x^2-25x \end{array}$$

Step 4: Bring down the next term $-25x$ from the previous dividend. $-3x^2-25x$ becomes the new dividend. **Next, the steps are repeated**.

$$\begin{array}{r} x^2 \; \boxed{-3x} \\ \boxed{x}+5 \overline{\big) x^3+2x^2-25x-50} \\ \underline{-x^3 - 5x^2} \\ \boxed{-3x^2}-25x \end{array}$$

Step 1: The 1^{st} term of the new dividend $-3x^2$ is divided by x, the 1^{st} term of the divisor, to arrive at $\boxed{-3x}$.

$$\boxed{-3x}(x+5) = -3x^2-15x$$

Step 2: The term $\boxed{-3x}$, from Step 1 is multiplied by the divisor.

$$\begin{array}{r} x^2-3x \\ x+5 \overline{\big) x^3+2x^2-25x-50} \\ \underline{-x^3 - 5x^2} \\ -3x^2-25x \\ \underline{\boxed{3x^2+15x}} \\ -10x \end{array}$$

Step 3: The result from Step 2 is subtracted, so it becomes $3x^2+15x$, as shown.

$$\begin{array}{r} x^2-3x \\ x+5 \overline{\big) x^3+2x^2-25x-50} \\ \underline{-x^3 - 5x^2} \\ -3x^2-25x \\ \underline{3x^2+15x} \\ -10x-50 \end{array}$$

Step 4: Bring down the next term -50 from the previous dividend. $-10x-50$ becomes the new dividend. **Next, the steps are repeated**.

$$\begin{array}{r}
x^2-3x\ \boxed{-10} \\
\boxed{x}+5\overline{\big)\,x^3+2x^2-25x-50} \\
\underline{-x^3-5x^2}\qquad\qquad\quad \\
-3x^2-25x\qquad\quad \\
\underline{3x^2+15x}\qquad\quad \\
\boxed{-10x}-50
\end{array}$$

Step 1: The 1^{st} term of the dividend $-10x$ is divided by x the 1^{st} term of the divisor, to arrive at $\boxed{-10}$.

$$-10(x+5) = -10x-50$$

Step 2: The term -10, from Step 1 is multiplied by the divisor.

$$\begin{array}{r}
\boxed{x^2-3x-10} \\
x+5\overline{\big)\,x^3+2x^2-25x-50} \\
\underline{-x^3-5x^2}\qquad\qquad\quad \\
-3x^2-25x\qquad\quad \\
\underline{3x^2+15x}\qquad\quad \\
-10x-50 \\
\underline{\boxed{10x+50}} \\
0
\end{array}$$

Step 3: The result from Step 2 is subtracted, so it becomes $10x+50$, as shown. This subtraction leads to zero. So, the remainder is zero. The quotient or final answer is $x^2-3x-10$.

One should keep in mind that long division is really just a matter of four steps that are repeated, over and over again.

Solved Problem 4.14 Solve $(6x^3+11x^2-31x+15)\div(3x-2)$ with long division.

The expression is rewritten as

$$3x-2\overline{\big)\,6x^3+11x^2-31x+15}$$

$\frac{\boxed{6x^3}}{3x} = 2x^2$ leads to the first term in the solution. Multiplying $2x^2$ by the divisor $3x-2$ leads to $6x^3-4x$. When this binomial is subtracted it becomes $\boxed{-6x^3+4x}$. Subtraction and bringing down the next term in the dividend leads to $\boxed{15x^2-31x}$. Next the steps repeat.

$$\begin{array}{r}
2x^2\qquad\qquad\qquad \\
3x-2\overline{\big)\,\boxed{6x^3}+11x^2-31x+15} \\
\underline{-6x^3\ +\ 4x^2}\qquad\qquad\quad \\
\boxed{15x^2-31x}\qquad
\end{array}$$

$\dfrac{\boxed{15x^2}}{3x} = 5x$ leads to the second term in the solution $5x$. Multiplying $5x$ by the divisor $3x-2$ leads to $15x^2-10$. When this binomial is subtracted it becomes $\boxed{-15x^2+10}$. Subtraction and bringing down the next term in the dividend leads to $\boxed{-21x+15}$.

$$\begin{array}{r} 2x^2+5x \\ 3x-2\overline{)6x^3+11x^2-31x+15} \\ \underline{-6x^3+\ 4x^2} \\ \boxed{15x^2}-31x \\ \underline{-15x^2\ +10x} \\ \boxed{-21x+15} \end{array}$$

$\dfrac{\boxed{-21x}}{3x} = -7$ leads to the third term in the solution -7. Multiplying -7 by the divisor $3x-2$ leads to $-21x+14$. When this binomial is subtracted it becomes $\boxed{21x-14}$. Subtraction leads to the remainder of $\boxed{1}$.

$$\begin{array}{r} \boxed{2x^2+5x-7} \\ 3x-2\overline{)6x^3+11x^2-31x+15} \\ \underline{-6x^3+\ 4x^2} \\ 15x^2-31x \\ \underline{-15x^2+10x} \\ \boxed{-21x}+15 \\ \underline{21x\ -14} \\ \boxed{1} \end{array}$$

$$\boxed{\text{Final Answer } 2x^2+5x-7 \text{ Remainder } 1 \text{ or } 2x^2+5x-7+\frac{1}{3x-2}}$$

An alternative approach to polynomial division is synthetic division. Synthetic division makes it possible to divide a polynomial by a linear binomial. Synthetic division can be applied when one is dividing by a **linear** binomial in the form $(x-c)$. The following will be necessary to carry out synthetic division.

1. The zero of the linear binomial, or for the linear binomial in the form $(x-c)$, the constant c.
2. All the coefficients of the polynomial, including the coefficients of the missing terms.

One can make the following observations about the binomial, $(x-c)$.

- It is linear
- The exponent of x is 1
- c is a constant
- The constant is subtracted

Linear functions were presented in a visual way in Solved Problem 1.2. The equations of linear and nonlinear functions were further contrasted on page 80. A key point is that for a linear binomial, the variable of x will simply have an exponent of 1. Examples of linear binomials are $(x-5)$, $(x-8)$, and $(x+12)$.

Synthetic division requires one to divide by a linear binomial in the form $(x-c)$, such as the binomial $(x-2)$. However, notice that $+2 = -(-2) = +2$. For this reason $(x+2) = (x-(-2))$. So, if the binomial $(x+2)$ is placed in the form of "x minus something", as in $(x-(-2))$, the constant will be negative, as shown. So with synthetic division one can divide by $(x+10)$, $(x+3)$, or $(x+9)$, but these binomials must be converted to the form $(x-c)$.

Solved Problem 4.15 Convert these linear, binomials to the form $(x-c)$.

(a) $(x+15)$ (b) $(x+7)$ (c) $(x+24)$ (d) $(x+2)$

(a) $+15 = -(-15)$ so $(x+15) = \boxed{\text{Final Answer } (x-(-15))}$ c is then -15.

(b) $+7 = -(-7)$ so $(x+7) = \boxed{\text{Final Answer } (x-(-7))}$ c is then -7.

(c) $+24 = -(-24)$ so $(x+24) = \boxed{\text{Final Answer } (x-(-24))}$ c is then -24.

(d) $+2 = -(-2)$ so $(x+2) = \boxed{\text{Final Answer } (x-(-2))}$ c is then -2. ■

Notice in the last solved problem that for all four binomials finding the constant c is the same as finding the zero of each binomial. If each binomial is set equal to zero, then the value of x, also called the zero of the binomial, will be the value of c that is needed.

What are the coefficients in the polynomial $x^4 - 7x^2 - 6x$? The polynomial has a degree of 4. Yes, the coefficients of $x^4 - 7x^2 - 6x$ are 1, -7, and -6. One can also say that in the polynomial $x^4 - 7x^2 - 6x$, there is a term with x^3, but with a coefficient of zero. Likewise, for the term with x^0, the coefficient is, again, zero. The terms with x^3 and x^0, are said to be missing terms, in this polynomial. The coefficients of the missing terms are simply zero. The following expression then shows the coefficients of all the missing terms.

$$x^4 - 7x^2 - 6x = x^4 \boxed{+0x^3} -7x^2 - 6x \boxed{+0x^0}$$

The polynomial has been expanded to include all the exponents up until 4, which is the degree of the polynomial.

Solved Problem 4.16 For each polynomial, list all the **coefficients**, including the coefficients for the missing terms.

(a) $x^3 + 7$ (b) $5x^4$ (c) $10x^3 - 2$ (d) $7x^4 - x^2$

The coefficients of the missing terms are simply zero! Find the missing terms and set the coefficients to zero.

(a) $x^3+7 = 1x^3 + \boxed{+0x^2}\boxed{+0x^1} + 7x^0$ $\boxed{\text{Final Answer 1, 0, 0, 7}}$

(b) $5x^4 = 5x^4 \boxed{+0x^3}\boxed{+0x^2}\boxed{+0x^1}\boxed{+0x^0}$ $\boxed{\text{Final Answer 5, 0, 0, 0, 0}}$

(c) $10x^3-2 = 10x^3 \boxed{+0x^2}\boxed{+0x^1} -2$ $\boxed{\text{Final Answer 10, 0, 0, -2}}$

(d) $7x^4-x^2 = 7x^4 \boxed{+0x^3} -x^2 \boxed{+0x^1}\boxed{+0x^0}$ $\boxed{\text{Final Answer 7, 0, -1, 0, 0}}$ ▪

In order to divide the polynomial $\left(x^4-7x^2-6x\right)$ by the binomial $x+2$, with synthetic division.

1. First, the linear binomial is rewritten as $x-(-2)$. Notice that the constant c, becomes -2. Also, -2 is the zero of the binomial $x+2$.
2. All the coefficients are listed, including the coefficients of the missing terms. All the coefficients are $\boxed{\text{1, 0, -7, 6, 0}}$

Now, synthetic division can be carried out. The next steps are demonstrated below.

$$\frac{x^4-7x^2-6x}{x+2} \quad \text{is rewritten as} \quad \begin{array}{r|rrrrr} -2 & 1 & 0 & -7 & -6 & 0 \\ & & & & & \\ \hline \end{array}$$

The first coefficient is brought down as shown. The zero -2 will be hereafter multiplied by every number that goes below the horizontal line.

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & -7 & -6 & 0 \\ & \downarrow & & & & \\ \hline & 1 & & & & \end{array}$$

The zero, -2 is multiplied by 1. So $-2(1) = -2$, which is placed in the next column above the horizontal line.

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & -7 & -6 & 0 \\ & & -2 & & & \\ \hline & 1 & & & & \end{array}$$

In the next step $0-2=-2$ which goes below the horizontal line.

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & -7 & -6 & 0 \\ & & -2 & & & \\ \hline & 1 & -2 & & & \end{array}$$

Again -2, is multiplied by the next number below the horizontal line, $-2(-2)=4$. Then 4, goes in the next column above the horizontal line. Then $-7+4=-3$, so 3 goes below the horizontal line.

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & -7 & -6 & 0 \\ & & -2 & 4 & & \\ \hline & 1 & -2 & -3 & & \end{array}$$

Again -2, is multiplied by the next number below the horizontal line, $-2(-3) = 6$. Then 6 goes in the next column above the horizontal line. Then $-6+6=0$, so zero goes below the horizontal line.

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & -7 & -6 & 0 \\ & & -2 & 4 & 6 & \\ \hline & 1 & -2 & -3 & 0 & \end{array}$$

Again, -2, is multiplied by the next number below the horizontal line, $-2(0) = 0$. Then 0, goes in the next column above the horizontal line. Then $0+0=0$, so zero goes below the horizontal line.

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & -7 & -6 & 0 \\ & & -2 & 4 & 6 & 0 \\ \hline & 1 & -2 & -3 & 0 & 0 \end{array}$$

Just, above, one can see the numbers, 1, -2, -3, 0, 0, just below the horizontal line. These numbers are the coefficients of the solution, except for the last number, which is the remainder. For this solution the remainder is zero. The degree of the polynomial dividend was 4, and **this means that the degree of the solution will be 1 less, which would be 3**. This means that the solution will have a term with x^3, with the form shown below, where only the coefficients need to be filled in.

$$\boxed{}\, x^3 + \boxed{}\, x^2 + \boxed{}\, x^1 + \boxed{}\, x^0 \quad \text{leads to} \quad \boxed{1}\, x^3 + \boxed{-2}\, x^2 + \boxed{-3}\, x^1 + \boxed{0}\, x^0 \text{ R=0}$$

If the degree of the solution is known, and the coefficients of the solution are known, then the full solution can be assembled, as shown above, to the right.

Solved Problem 4.17 Solve $\dfrac{2x^3 - 3x^2 + 4x + 5}{x+2}$.

The divisor is the linear binomial $x+2$. It fits the form $x-c$, if $x+2 = x-(-2)$ and $\boxed{c=-2}$. Also, if $x+2$ is set equal to zero, as in $x+2=0$, then $x=-2$. As mentioned, c is equal to the zero of the binomial divisor.

Then all the coefficients of the dividend $2x^3 - 3x^2 + 4x + 5$, including the coefficients for missing terms are shown below.

$$2x^3 - 3x^2 + 4x + 5 = \boxed{2}\, x^3 \boxed{-3}\, x^2 + \boxed{4}\, x^1 + 5 \quad \text{The coefficients are } \boxed{2,\ -3,\ 4,\ 5}.$$

The first coefficient is brought down as shown. The zero, $\boxed{-2}$, is hereafter multiplied by every number that goes below the horizontal line.

$$\begin{array}{r|rrrr} -2 & 2 & -3 & 4 & 5 \\ & \downarrow & & & \\ \hline & 2 & & & \end{array}$$

-2, is multiplied by the next number below the horizontal line, $-2(2) = -4$. Then $\boxed{-4}$, goes in the next column above the horizontal line. Then $-3 + -4 = -7$, so $\boxed{-7}$ goes below the horizontal line.

$$\begin{array}{r|rrrr} -2 & 2 & -3 & 4 & 5 \\ & & -4 & & \\ \hline & 2 & -7 & & \end{array}$$

Again -2, is multiplied by the next number below the horizontal line, $-2(-7) = 14$. Then $\boxed{14}$, goes in the next column above the horizontal line. Then $4 + 14 = 18$, so $\boxed{18}$ goes below the horizontal line.

$$\begin{array}{r|rrrr} -2 & 2 & -3 & 4 & 5 \\ & & -4 & 14 & \\ \hline & 2 & -7 & 18 & \end{array}$$

Again, -2 is multiplied by the next number below the horizontal line, $-2(18) = -36$. Then $\boxed{-36}$, goes in the next column above the horizontal line. Then $5 - 36 = -31$, so $\boxed{-31}$ goes below the horizontal line.

$$\begin{array}{r|rrrr} -2 & 2 & -3 & 4 & 5 \\ & & -4 & 14 & -36 \\ \hline & 2 & -7 & 18 & -31 \end{array}$$

The original dividend $2x^3 - 3x^2 + 4x + 5$ had a degree of 3, so the solution will have a degree of 2, and the form shown below. Just above the row below the horizontal line provides the coefficients, and the last number in the row, -31 is the remainder.

$\boxed{}\, x^2 + \boxed{}\, x^1 + \boxed{}\, x^0$ leads to $\boxed{\text{Final Answer } 2x^2 - 7x + 18 \quad \text{R}=-31}$ ■

Both long division and synthetic division have been demonstrated.

Definition 4.7 — Remainder Theorem. If the polynomial $f(x)$ is divided by the linear binomial $x - c$, then $f(c)$ is equal to the remainder.

A good way to describe the remainder theorem, would be with the last solved problem. In the last solved problem the polynomial $f(x) = 2x^3 - 3x^2 + 4x + 5$ is divided by the binomial $x + 2$. The remainder theorem calls for a binomial in the form $x - a$, so again the binomial is placed in the form $x - (-2)$, then $c = -2$. Synthetic division showed that the remainder was $\boxed{\text{-31}}$.

$$f(-2) = 2(-2)^3 - 3(-2)^2 + 4(-2) + 5 = 2(-8) - 3(4) - 8 + 5 = -16 - 12 - 8 + 5 = \boxed{-31}$$

4.6 Graphing Higher Order Polynomials

With an unknown polynomial, there is the need to know:

- Is a value increasing or decreasing?

- Where is a value increasing?
- Is a value increasing in a linear or straight manner?
- Is a value increasing in a nonlinear or curved manner?
- Are the values negative or positive (domain and range)?
- Will the graph display symmetry?

Quadratics were discussed in a visual and algebraic way in Chapter 2, and in Definition 2.10. A key characteristic with a quadratic equation is that the exponent on the variable x is 2, and that the degree is 2. Nonlinear functions were discussed on page 80. The parabola formed by $f(x) = x^2$ was demonstrated and analyzed. For a parabola, the axis of symmetry was defined in Definition 2.6, and its vertex was defined in 2.7. This chapter has introduced polynomials where the degree exceeds 2, as higher order polynomials. The graph of higher order polynomials, will often not be immediately obvious. This doesn't make a polynomial difficult. This means that some tools have to be used to better understand the polynomial. These tools are not difficult either. These are simply ways of describing lines and curves. How can one start to understand an unknown polynomial? How can one describe lines and curves?

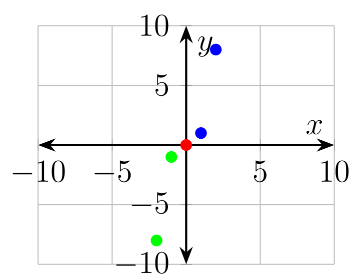
An unknown system can be analyzed by observing its input and output. On page 81, a parabola was first introduced by simply plotting a few points. This same tool makes it possible to begin to understand any unknown polynomial. For example, the function $f(x) = x^3$, also called a cubic function is not obvious at first. Recall that a function has an input, output nature. Plotting a few points will make it possible to see the "unknown".

- If $x = -2, y = -8$
- If $x = -1, y = -1$
- If $x = 1, y = 1$
- If $x = 2, y = 8$

Additional observations

- The graph crosses the origin (red point)
- Some y values are negative (green points)
- Some y values are positive (blue points)
- It is nonlinear (simply not a straight line)

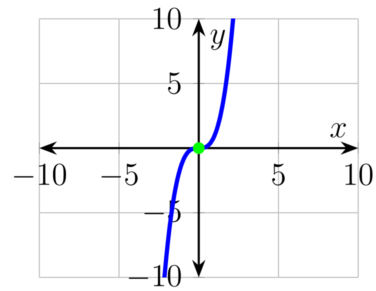
Fig. 4.2: $f(x) = x^3$



In Figure 4.2 above, and Figure 4.3 below, one can see that the graph is not symmetric. Yes, one could say that it is clearly different than the graph of $f(x) = x^2$. Whether a function is odd or even, would communicate this difference, in a clear way.

Plotting a few points reveals information about $f(x) = x^3$. If one continued plotting points one would see the smooth, blue curve, shown here, to the right. One can see that the curve increases when x is greater than zero. This curve decreases if x is less than zero.

Fig. 4.3: A Cubic Function



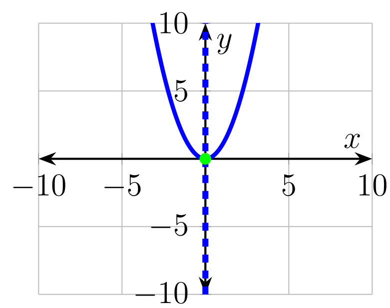
Definition 4.8 — Even Function. A function is even if it satisfies the condition $f(x) = f(-x)$. An even function displays symmetry about the y-axis.

Definition 4.9 — Odd Function. An odd function is odd if it satisfies the condition $f(-x) = -f(x)$. An odd function will remain unchanged if it rotates 180 degrees about the origin, and this is called rotational symmetry.

$f(x) = x^2$ is an Even Function

$f(1) = 1$ and $f(-1) = 1$
$f(3) = 9$ and $f(-3) = 9$
The condition $f(x) = f(-x)$ is satisfied
The axis of symmetry is shown as a blue dashed line.

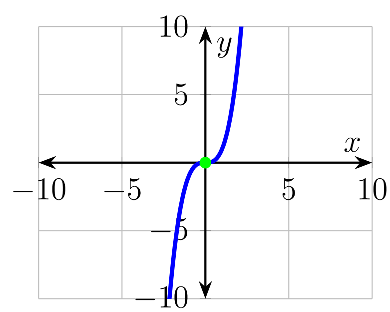
Fig. 4.4: $f(x) = x^2$



$f(x) = x^3$ is an Odd Function

$f(1) = 1$ and $f(-1) = -1$
$f(3) = 27$ and $f(-3) = -27$
The condition $f(-x) = -f(x)$ is satisfied

Fig. 4.5: $f(x) = x^3$



The symmetry in Figure 4.4 is easy to find. In Figure 4.4 the blue, dashed line shows the axis of symmetry. Notice in Figure 4.5, that if the function is rotated 180 degrees that it will look the same. This symmetry in Figure 4.5 is called rotational symmetry. In this case the symmetry does not occur about an axis. Instead, the symmetry is said to be about the origin. With rotational symmetry one needs to check if a function will look the same, after rotating it 180 degrees. This rotation is the same as a regular wall clock, where the hour hand rotates from 12 o'clock to 6 o'clock.

The function $f(x) = x^3$ was described conceptually, in plain English. Graphing a few points made it possible to better understand this function. It can now be described with more clarity.

- It is an odd function, with rotational symmetry.
- It crosses the origin
- Its Domain is $(-\infty, +\infty)$
- Its Range is also $(-\infty, +\infty)$
- It is nonlinear (simply not a straight line)

Solved Problem 4.18 Are the following functions odd, even, or neither?

(a) $f(x) = x^3 - 2x$
(b) $f(x) = x^2 - x^3 + 4x$
(c) $f(x) = x^2 - 18$
(d) $f(x) = x^4 - 8x^2 + 16$

(a) For $f(x) = x^3 - 2x \quad f(-x) = (-x)^3 - 2(-x)$

$$f(-x) = (-1)^3(x)^3 + 2x = -1x^3 + 2x = -(x^3 - 2x)$$

$$\boxed{\text{Since } f(-x) = -f(x) \text{ this is an odd function}}$$

(b) For $f(x) = x^2 - x^3 + 4x \quad f(-x) = (-x)^2 - (-x)^3 + 4(-x)$

$$f(-x) = (-1)^2x^2 - (-1)^3x^3 - 4x = 1x^2 + 1x^3 - 4x$$

$$\boxed{\text{Since } f(-x) \neq f(x) \text{ and } f(-x) \neq -f(x) \text{ this is neither even nor odd}}$$

(c) For $f(x) = x^2 - 18 \quad f(-x) = (-x)^2 - 18 = (-1)^2x^2 - 18 = 1x^2 - 18$

$$\boxed{\text{Since } f(-x) = f(x) \text{ this is an even function}}$$

(d) For $f(x) = x^4 - 8x^2 + 16$

$$f(-x) = (-x)^4 - 8(-x)^2 + 16 = (-1)^4x^4 - 8(-1)^2x^2 + 16 = 1x^4 + 8x^2 + 16$$

$$\boxed{\text{Since } f(-x) = f(x) \text{ this is an even function}}$$

Evaluating polynomials can be carried out just as it was for the function $f(x) = x^3$.

$$f(x) = 7x^2 - 3x + 2 \text{ then } \boxed{f(-2)} = 7(-2)^2 - 3(-2) + 2 = 7(4) + 6 + 2 = \boxed{36}$$

$$f(x) = 3x^2 + 2x + 5 \text{ then } \boxed{f(4)} = 3(4)^2 + 2(4) + 5 = 48 + 13 = \boxed{61}$$

For quadratic equations the points at which the equation is equal to zero, have been called zeros, roots, or solutions. This is the same as saying that the x-intercepts of the function have been called zeros, roots, or solutions. It was emphasized that a quadratic equation can have zero solutions, one solution, or two solutions.

With a polynomial, one is often interested in the points where the equation is equal to zero. Polynomial equations, where the degree is higher than two, can have more than 2 solutions. The rational root theorem can be used to find the rational solutions.

Definition 4.10 — Rational Root Theorem. For a polynomial with a constant and a leading coefficient, rational solutions can be written in the form $\frac{p}{q}$. Then p is a factor of the constant, and q is a factor of the leading coefficient. In other words, factors of the constant term, and factors of the leading coefficient can be used to find rational solutions.

Based on the definition of the rational root theorem, one can find possible rational solutions to a polynomial by factoring:

- The leading coefficient
- The constant term

Once possible solutions are found, the Remainder Theorem can be used to check and confirm solutions. The leading coefficient was discussed on page 137. While learning to solve a polynomial with the Rational Root Theorem the following steps can be helpful.

Step 1 Factor the constant
Step 2 Factor the leading coefficient
Step 3 Combine the factors in rational form $\frac{p}{q}$ to list all the possible solutions
Step 4 Test possible solutions.

These steps repeat until the roots are identified. These steps are demonstrated below for the polynomial $\boxed{3x^3 - 4x^2 - 17x + 6}$

Step 1 Factor the constant

The constant is 6. The list of factors of 6 labeled as p includes $\pm 1, \pm 2, \pm 3, \pm 6$.

Step 2 Factor the leading coefficient

The leading coefficient is 3. The list of factors of 3 labeled as q includes $\pm 1, \pm 3$.

Step 3 Combine the factors in rational form $\frac{p}{q}$ to list all the possible solutions

The factors in Step 1 and Step 2 can be combined in an orderly way. For example, first combine ± 1 in q with all the factors in p. Then again, combine ± 3 in q with all the factors in p.

For $q = \pm 1$ $\quad \dfrac{p}{q} = \dfrac{\pm 1}{\pm 1}, \dfrac{\pm 2}{\pm 1}, \dfrac{\pm 3}{\pm 1}, \dfrac{\pm 6}{\pm 1} = \pm 1, \pm 2, \pm 3, \pm 6$ $\quad$ so $\boxed{\dfrac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6}$

Then for $q = \pm 3$ $\quad \boxed{\dfrac{p}{q} = \dfrac{\pm 1}{\pm 3}, \dfrac{\pm 2}{\pm 3}, \dfrac{\pm 3}{\pm 3}, \dfrac{\pm 6}{\pm 3}}$

These 2 lists of possible roots in the form $\dfrac{p}{q}$ have duplicates because $\dfrac{\pm 3}{\pm 3} = \pm 1$ and $\dfrac{\pm 6}{\pm 3} = \pm 2$

so after removing duplicates $\boxed{\dfrac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6, \dfrac{\pm 1}{\pm 3}, \dfrac{\pm 2}{\pm 3}}$

Step 4 Test possible solutions

According to the Remainder Theorem, if a root divides a polynomial then the remainder will be zero. So one tests each possible root $\dfrac{p}{q}$. This step becomes an exercise in synthetic division.

Test the possible root -1

$$\begin{array}{r|rrrr} -1 & 3 & -4 & -17 & 6 \\ & \downarrow & -3 & 7 & 10 \\ \hline & 3 & -7 & -10 & \boxed{16} \end{array}$$

Remainder is 16, not a solution

Test the possible root -2

$$\begin{array}{r|rrrr} -2 & 3 & -4 & -17 & 6 \\ & \downarrow & -6 & 20 & -6 \\ \hline & 3 & -10 & 3 & \boxed{0} \end{array}$$

Remainder is 0, it is a solution

Test the possible root 3

$$\begin{array}{r|rrrr} 3 & 3 & -4 & -17 & 6 \\ & \downarrow & 9 & 15 & -6 \\ \hline & 3 & 5 & -2 & \boxed{0} \end{array}$$

Remainder is 0, it is a solution

Three possible solutions are tested above with synthetic division. This revealed two solutions. In the same way, one would test all the possible solutions. This would confirm that the solutions of this polynomial are $\boxed{x = -2,\ x = \dfrac{1}{3},\ \text{and } x = 3}$.

Above, one can see that one might have to test 12 possible solutions, since one needs to check each positive and each negative solution. The degree of a polynomial is a time saver when checking possible solutions.

Tip The degree of a polynomial is also the maximum number of solutions

The polynomial $3x^3-4x^2-17x+6$ has a degree of three, so it has a maximum of 3 solutions. This can save time when testing the possible solutions.

Solved Problem 4.19 Rational Root Theorem - Solve $x^3-x^2-10x-8=0$

The leading coefficient and the constant are identified in the equation

$$\boxed{1}\ x^3-x^2-10x\ \boxed{-8}=0$$

Step 1 Factor the constant

The constant is -8. Factors p are the factors of 8 $\pm1,\pm2,\pm4,\pm8$

Step 2 Factor the leading coefficient

The leading coefficient is 1. Factors q are the factors of 1 ±1

Step 3 Combine the factors in rational form $\frac{p}{q}$ to list all the possible solutions

For factors $q=\pm1$ the list of **possible** solutions $\frac{p}{q}=\frac{\pm1}{\pm1},\frac{\pm2}{\pm1},\frac{\pm4}{\pm1},\frac{\pm8}{\pm1}$

Step 4 Test possible solutions

Test the possible root -1
$$\begin{array}{r|rrrr} -1 & 1 & -1 & -10 & -8 \\ & \downarrow & -1 & 2 & 8 \\ \hline & 1 & -2 & -8 & 0 \end{array}$$
Remainder is 0, it is a solution

Test the possible root 1
$$\begin{array}{r|rrrr} 1 & 1 & -1 & -10 & -8 \\ & \downarrow & 1 & 0 & -10 \\ \hline & 1 & 0 & -10 & -18 \end{array}$$
Remainder is -18, not a solution

Test the possible root -2
$$\begin{array}{r|rrrr} -2 & 1 & -1 & -10 & -8 \\ & \downarrow & -2 & 6 & 8 \\ \hline & 1 & -3 & -4 & 0 \end{array}$$
Remainder is 0, it is a solution

The degree of this polynomial is 3. Two solutions -1 and -2 have been found. Only one more solution needs to be found.

Other possible solutions would be tested with synthetic division as shown above to confirm the solutions of this polynomial. $\boxed{\text{Final Answer } -1,-2, \text{ and } 4}$ ■

The concept of multiplicity makes it possible to further describe the graph of a polynomial.

Definition 4.11 — Multiplicity. A solution or zero of a polynomial has an associated factor. The number of times that this factor appears, in the polynomial, is called multiplicity.

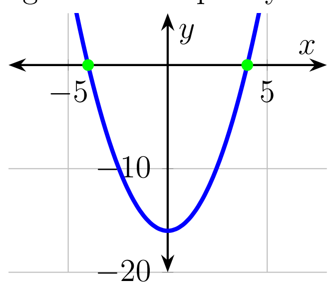
The multiplicity of a factor may be odd or even. This provides information about the nature of the polynomial, near the solution. The function $f(x) = x^2 - 16$ is graphed below.

$x^2 - 16 = \boxed{(x-4)}\,\boxed{(x+4)}$

The solutions are $x = -4,\ 4$. The associated factors are $(x-4)(x+4)$

For -4, the associated factor appears once. Its multiplicity is 1. For 4, the associated factor appears once. Its multiplicity is 1.

Fig. 4.6: Multiplicity = 1



In the graph, just above, notice that for the root -4, that the polynomial crosses the x-axis. If a solution has odd multiplicity the polynomial will cross the x-axis. Now a polynomial can cross the x-axis in varying ways. In the graph above, the polynomial passes straight through the x-axis. How else might a polynomial cross the x-axis?

Tip If a solution has odd multiplicity, the polynomial will cross the x-axis at that point.

In the graph below, the multiplicity of 3 is higher than one, but the multiplicity is again odd. Again the polynomial crosses the x-axis at the solution. However, notice that it crosses the x-axis, in a different way, than the graph above. In the graph below, the polynomial curve seems to flex and flatten at the x-axis, but it still crosses the x-axis.

This graph shows $f(x) = x^3$

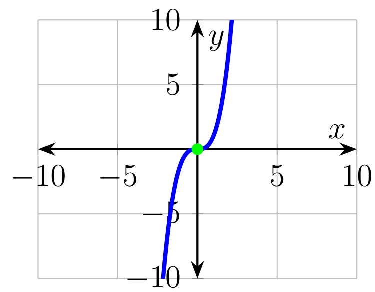
For this monomial, the solution is zero, and the associated factor is x

This factor has an exponent of 3

The factor appears 3 times

The multiplicity is 3

Fig. 4.7: Multiplicity = 3



What happens if a solution has an even multiplicity?

$$f(x) = x(x-4)^3(x+3)^2$$

The polynomial just above is written in terms of its factors. The factor x will lead to the solution of zero, and its associated factor x has a multiplicity of 1, because this associated factor appears only once. The exponent of the factor $(x-4)$ makes it clear that this factor appears 3 times, so for the associated solution $x = 4$ the multiplicity is 3. Likewise, the exponent of the factor $(x+3)^2$ makes it clear that for the solution $x = -3$ the multiplicity is 2, an even multiplicity. The point (-3, 0) is shown, just below, in green. One can see that at this point the polynomial touches the x-axis, but it does not cross the x-axis. The polynomial turns back or turns around at this point. This is characteristic for an even multiplicity. This is further demonstrated, just below.

$f(x) = x(x-4)^3(x+3)^2$

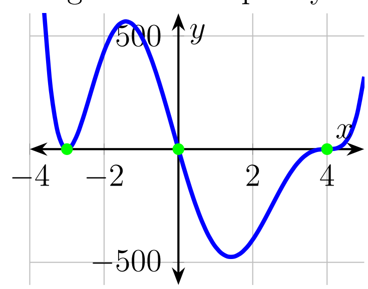
The solutions are 0, 4, and -3

Solution 0, multiplicity is 1, odd.

Solution 4, multiplicity is 3, odd.

Solution -3, multiplicity is 2, even.

Fig. 4.8: Multiplicity



In the graph, just above, for the solution $x = 4$, the polynomial displays a flattening behavior at $x = 4$, as expected, even though it does cross the x-axis.

Tip If a solution has even multiplicity, the polynomial will turn around at that point, and it will not cross the x-axis.

4.7 Higher Degree Polynomials Problems

Click on the page number to the right to see the chapter discussion. Click on the solution number to navigate to the solution.

Convert the following decimals to percentages.

1. (a) .45 (b) .1 (c) .022 (Page 134) (Solution 1)

2. (a) 3 (b) 2.5 (c) .019 (Page 134) (Solution 2)

Rewrite these percentages in decimal form.

3. (a) 12% (b) 9% (c) 3% (Page 134) (Solution 3)

4. (a) 120% (b) 215% (c) 180% (Page 134) (Solution 4)

5. Adam lends $50,000$ U.S. Dollars (USD) to Brian at an interest rate of 5% for three years. What is the simple interest for this loan? (Page 135) (Solution 5)

6. Eve lends $62,000$ U.S. Dollars (USD) to Belinda at an interest rate of 3% for 4 years. What is the simple interest for this loan? (Page 135) (Solution 6)

7. Interest is paid at 2%, twice per year on a principal amount of $15,000$ USD, for 3 years. Find the final amount. (Page 136) (Solution 7)

8. For a principal amount of 100,000 USD, interest is paid at 3.2% every 6 months for 10 years. Find the final amount. (Page 136) (Solution 8)

9. Which of the following expressions are polynomials? (Page 137) (Solution 9)

   (a) $\frac{1}{2x+5}$ (b) x^3+x^2+1 (c) $12x^3$ (d) $x^{-4}+2x+7$

10. Which of the following expressions are polynomials? (Page 137) (Solution 10)

    (a) $2x^7+3x^2+9$ (b) $\frac{1}{2x}+5x^2+\frac{3}{5x}$ (c) x^5 (d) $x^{-1}+13x^2+6$

11. For $f(x)=3x^2-9x+7$ evaluate $f(3)$. (Page 139) (Solution 11)

12. For $f(x)=4x^2+3x+5$ evaluate $f(-1)$. (Page 139) (Solution 12)

13. Add the polynomials $3x^3-5x+9$ and $6x^3+8x-4$. (Page 140) (Solution 13)

14. Add the polynomials $3x^2+5x+2$ and $4x^2-6x+8$. (Page 141) (Solution 14)

15. Solve (x^2-3) - $(2x^2+4x-6)$. (Page 141) (Solution 15)

16. Solve $(9x^2-9x+19)$ - $(-2x^2+3x-2)$. (Page 141) (Solution 16)

17. Solve $(x+5)(x+8)$.
    (Page 142) (Solution 17)

18. Solve $(x+3)(x^2-4)$.
    (Page 142) (Solution 18)

19. Solve $(3x+6)(5x^2+3x-10)$.
    (Page 142) (Solution 19)

20. For each polynomial, list all the coefficients, including the coefficients for the missing terms. (a) $3x^5+x^2$ (b) $9x^4+12x+8$ (Page 147) (Solution 20)

21. Long Division

    Solve $\frac{27x^3+9x^2-3x-10}{3x-2}$.
    (Page 145) (Solution 21)

22. Long Division

    Solve $\frac{6x^3-8x+5}{2x-4}$.
    (Page 145) (Solution 22)

23. Synthetic Division

    Divide $\left(x^2+2x+6\right)$ by $(x-1)$.
    (Page 149) (Solution 23)

24. Synthetic Division

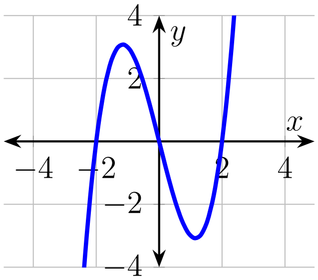
    Solve $\frac{x^3+2x^2-25x-50}{x+5}$.
    (Page 149) (Solution 24)

25. Are the following functions odd, even, or neither?

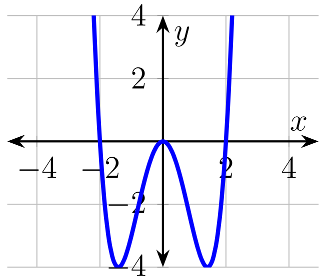
    (a) $f(x)=x^4-4x^2$ (b) $f(x)=x^3-5x^2-x+5$ (Page 153) (Solution 25)

26. Are the following functions odd, even, or neither? (Page 153) (Solution 26)

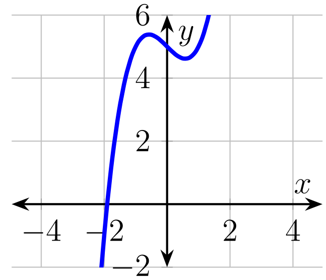
Graph A



Graph B



Graph C



27. Find rational roots for $x^3 - 5x^2 + 2x + 8 = 0$ (Page 154) (Solution 27)

28. Find rational roots for $x^3 + 2x^2 - x - 2 = 0$ (Page 154) (Solution 28)

29. For the polynomial written in terms of its factors $f(x) = (x-3)(x-1)^2$ (Page 157) (Solution 29)
    (a) What is the multiplicity of each solution?
    (b) At what point does the polynomial cross the x-axis?
    (c) At what point does the polynomial touch the x-axis and turn around?
    (d) At $x = 0$ is y positive or negative?
    (e) At $x = 2$ is y positive or negative?
    (f) At $x = 4$ is y positive or negative?

30. For the polynomial written in terms of its factors $f(x) = (x+3)(x-2)^2(x+1)^3$ (Page 157) (Solution 30)
    (a) What is the multiplicity of each solution?
    (b) At what point does the polynomial cross the x-axis?
    (c) At what point does the polynomial touch the x-axis? and turn around?
    (d) At $x = -2$ is y positive or negative?
    (e) At $x = 0$ is y positive or negative?
    (f) At $x = 1$ is y positive or negative?

4.8 Higher Degree Polynomials Solutions

1. (a) 45% (b) 10% (c) 2.2%

2. (a) 300% (b) 250% (c) 1.9%

3. (a) .12 (b) .09 (c) .03%

4. (a) 1.2 (b) 2.15 (c) 1.8

5. 7,500 USD

6. 7,440 USD

7. 15,922.80 USD

8. 137,364.39 USD

9. Only (b) x^3+x^2+1

10. Only (a) $2x^7+3x^2+9$

11. $f(3)=7$

12. $f(-1)=6$

13. $9x^2+3x+5$

14. $7x^2-x+10$

15. $-x^2-4x+3$ or $-\left(x^2+4x-3\right)$

16. $11x^2-12x+21$

17. $x^2+13x+40$

18. $x^3+3x^2-4x-12$

19. $15x^3+39x^2-12x-60$

20. (a) 3, 0, 0, 1, 0, 0 (b) 9, 0, 0, 12, 8

21. $9x^2+9x+5$

22. The quotient is $3x^2+6x+8$, remainder is 37. The full answer is $3x^2+6x+8+\dfrac{37}{2x-4}$

23. The quotient is $x+3$ remainder is 9. The full answer is $x+3+\dfrac{9}{x-1}$.

24. $x^2-3x-10$

25. (a) even (b) neither

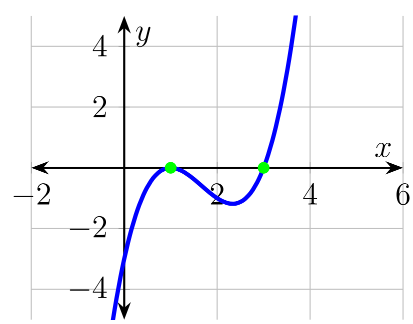
26. Graph A is odd, Graph B is even, Graph C is neither

27. x = -1, 2, 4

28. x = -2, -1, 1

29. $f(x)=(x-3)(x-1)^2$

(a) For $x=1$ multiplicity is 2
For $x=3$ multiplicity is 1
(b) $x=3$ (c) $x=1$
(d) Negative (e) Negative
(f) Positive



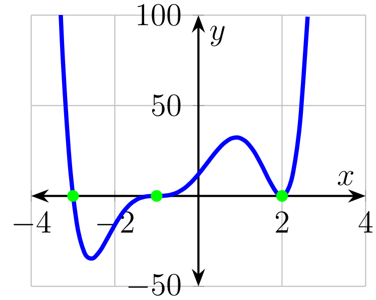
30. $f(x)=(x+3)(x-2)^2(x+1)^3$

(a) For $x = -3$ multiplicity is 1.
For $x = -1$ multiplicity is 3.
For $x = 2$ multiplicity is 2.

(b) $x = -3$ and $x = -1$

(c) $x = 2$ (d) Negative

(e) Positive (f) Positive



Chapter 5: Rational Functions

OVERVIEW

The sections of this chapter are:

5.1 Rational Expression Multiplication
5.2 Rational Expression Division
5.3 Rational Expression Addition
5.4 Rational Expression Subtraction
5.5 Rational Functions
5.6 Rational Function Inequalities

Learning rational functions helps you solve real-world problems, enhancing your problem-solving skills. It teaches you to think creatively and organize information effectively, which is useful in various fields. Rational functions are like systems with inputs and outputs, and examples of systems are everywhere in daily life. So, understanding rational functions can improve your ability to analyze and address challenges in different areas.

OBJECTIVES

By the end of the chapter, a student will be able to:

- Describe and discuss the input, output nature of a rational function
- Multiply, divide, add, and subtract rational expressions
- Recognize and describe rational expressions
- Understand complex rational expressions
- Combine the concepts of rational functions and inequalities

5.1 Rational Expression Multiplication

Rational in everyday language has a different meaning than rational in algebra. In algebra rational numbers simply refer to a ratio of numbers or values. One vehicle may travel twice as fast as another. A process may take four times as long as another. One house might have 3 floors, and another house might have one floor. These concepts can be expressed as ratios.

Definition 5.1 — Rational Expression. A ratio or a fraction where the numerator and the denominator are both polynomials or monomials is called a rational expression.

If house A has 1 floor, and house B has 3 floors, the number of floors would be a ratio.

$$\frac{\text{Floors in house A}}{\text{Floors in house B}} = \frac{1 \text{ Floor}}{3 \text{ Floors}} = \frac{1}{3}$$

If vehicle A has 4 wheels and vehicle B has 2 wheels, the number of wheels would be a ratio.

$$\frac{\text{Wheels on vehicle A}}{\text{Wheels on vehicle B}} = \frac{4\text{ Wheels}}{2\text{ Wheels}} = \frac{4}{2} = 2$$

Just above, one can see how a fraction is used to communicate a ratio of two values. In the same way an expression can communicate a ratio of two polynomials. Just below, three examples of rational expressions are shown.

$$\frac{x^2-1}{x^2+5} \qquad \frac{2x+5}{x^4-x^2} \qquad \frac{x+5}{2x-5}$$

Remember, numerator can't be zero in ratios. Above, both parts are binomials. Usually, they're polynomials, but monomials are okay. "Poly" means many, referring to 2+ terms. Some call even one term a polynomial. The ones below are also rational expressions.

$$\frac{1}{x^2-4} \qquad \frac{3x^2+5}{2} \qquad \frac{x-1}{x}$$

Notice in the expressions, just above, that 1 and 2 are constants. Still 1, 2, and x are monomials. These definitions both call for a non-negative, whole number exponent on the variable. For this reason, the following expressions are not rational expressions.

$$\frac{1}{\sqrt{x}-4} \qquad \frac{3x^{(1/3)}+5}{2} \qquad \frac{x-1}{x^{(1/2)}}$$

Recall, that $\sqrt{x} = x^{(1/2)}$ and $1/2$ is not a whole number. For this reason, in the expression $\sqrt{x}$ the exponent on x is not a whole number.

Solved Problem 5.1 Are the following expressions rational expressions?

(a) $\frac{4x}{20x^2-3}$ (b) $\frac{x^3+x^{(1/2)}}{x^2-5}$ (c) $\frac{x-2}{x^2-4x+16}$ (d) $\frac{x^2-16}{x^{(1/3)}}$

(a) $\frac{4x}{20x^2-3}$ $\boxed{\text{Yes, a rational expression}}$

(b) $\frac{x^3+x^{(1/2)}}{x^2-5}$ $\boxed{\text{Not a rational expression}}$

This is because the exponent $1/2$ is not a whole number.

(c) $\frac{x-2}{x^2-4x+16}$ $\boxed{\text{Yes, a rational expression}}$

(d) $\frac{x^2-16}{x^{(1/3)}}$ $\boxed{\text{Not a rational expression}}$

This is because the exponent 1/3 is not a whole number. ■

Simplifying rational expressions, brings into play working with exponents. Simplifying rational expressions should be practiced before multiplying rational expressions. This is demonstrated below.

$$\frac{6x^2}{12x^4} = \frac{6x^2}{2(6)x^2x^2} = \frac{\cancel{6}\cancel{x^2}}{2(\cancel{6})\cancel{x^2}x^2} = \frac{1}{2x^2}$$

Above, one can see that the coefficient 12 is expanded and rewritten in terms of its factors $12 = 2(6)$. Likewise, x^4 is rewritten in terms of its factors as $x^4 = x^2x^2$. Just above, the factors of 12 and the factors of x^4 make it possible to simplify this expression. Simplifying a rational expression is demonstrated again below.

$$\frac{(3x+1)(x+2)}{(x+2)} = \frac{(3x+1)\cancel{(x+2)}}{\cancel{(x+2)}} = 3x+1$$

Solved Problem 5.2 Simplify the following rational expressions.

(a) $\frac{15x^4}{3x^2}$ (b) $\frac{10x^8}{100x^3}$ (c) $\frac{(x+3)}{(x+3)(x+9)}$ (d) $\frac{(x+5)(x+6)}{x(x-5)(x+5)}$

(a) $$\frac{15x^4}{3x^2} = \frac{5\left(3x^2\right)x^2}{3x^2} = \frac{5\cancel{\left(3x^2\right)}x^2}{\cancel{\left(3x^2\right)}} = \frac{5x^2}{1} = \boxed{5x^2}$$

(b) $$\frac{10x^8}{100x^3} = \frac{10x^3x^5}{10(10x^3)} = \frac{\cancel{\left(10x^3\right)}x^5}{10\cancel{\left(10x^3\right)}} = \boxed{\frac{x^5}{10}}$$

(c) $$\frac{(x+3)}{(x+3)(x+9)} = \frac{\cancel{(x+3)}}{\cancel{(x+3)}(x+9)} = \boxed{\frac{1}{x+9}}$$

(d) $$\frac{(x+5)(x+6)}{x(x-5)(x+5)} = \frac{\cancel{(x+5)}(x+6)}{x(x-5)\cancel{(x+5)}} = \boxed{\frac{(x+6)}{x(x-5)} \text{ or } \frac{x+6}{x^2-5x}}$$ ■

This same concept of simplifying expressions can be used while multiplying polynomials. Now, consider the multiplication of rational expressions shown below.

$$\frac{x+3}{x^2+12x+27} \times \frac{x+9}{1}$$

In the multiplication shown above, one could proceed by multiplying the numerators, and by multiplying the denominators. In this case, the numerators are both binomials, and binomial multiplication was practiced in the previous chapter. There is another route and that would be, to first look for factors of each polynomial. For example, the factors of the denominator $x^2+12x+27$ are $(x+3)(x+9)$. Below, one can see how this is useful.

$$\frac{x+3}{x^2+12x+27}\times\frac{(x+9)}{1} = \frac{x+3}{(x+3)(x+9)}\times\frac{(x+9)}{1} = \frac{\cancel{(x+3)}}{\cancel{(x+3)}(x+9)}\times\frac{(x+9)}{1}$$

$$\frac{1}{(x+9)}\times\frac{(x+9)}{1} = \frac{1}{\cancel{(x+9)}}\times\frac{\cancel{(x+9)}}{1} = \frac{1}{1} = \boxed{1}$$

The factors were provided above, but this shows that factoring can really simplify the multiplication of rational expressions. For this reason, one should look for factors before multiplying rational expressions.

It may not be possible to further factor nor simplify a rational expression. In this case, the multiplication of rational expressions proceeds a lot like polynomial multiplication. First one multiplies the numerators. Then one multiplies the denominators. This is demonstrated, just below.

$$\frac{2}{(x-4)}\times\frac{3}{(x+1)} = \frac{2(3)}{(x-4)(x+1)} = \frac{6}{\boxed{x^2-3x-4}}$$

Polynomial multiplication is used to multiply the denominators.

$$\underset{1\quad 2}{\overset{1\quad 2}{(x-4)(x+1)}} = \overset{1\quad 2}{x(x+1)} + \underset{1\quad 2}{-4(x+1)} = \boxed{x^2-3x-4}$$

Solved Problem 5.3 Solve $12\times\dfrac{4x^3+x}{6x}$.

A first step is to multiply 12 by the numerator.

$$12\times\frac{4x^3+x}{(6)\,x} = \frac{2\,\boxed{(6)}\left(4x^3+x\right)}{\boxed{(6)}\,x} \qquad \frac{2\cancel{(6)}(x)\left(4x^2+1\right)}{\cancel{(6)}x} = \frac{2\cancel{(x)}\left(4x^2+1\right)}{\cancel{x}}$$

Notice that the numerator 12 and the denominator 6 have factors in common.

Factors of 12 are 1, 3, 4, $\boxed{6}$, 12. Factors of 6 are 1, 2, 3, $\boxed{6}$.

The factors of the numerator $4x^3+x$ are $x(4x^2+1)$.

$\boxed{\text{Final answer } 2\left(4x^2+1\right) = \text{or } 8x^2+2}$ ■

Solved Problem 5.4 Solve $\dfrac{4x-16}{18x-9} \times \dfrac{2x-1}{(x-4)^2}$.

One route could be to multiply the numerators and to multiply the denominators. A preferable route is to factor the numerators and denominators.

The numerator $4x-16$ has the factors $4(x-4)$

The denominator $18x-9$ has the factors $9(2x-1)$

The denominator $(x-4)^2$ has the factors $(x-4)(x-4)$.

The initial multiplication expression can be rewritten.

$$\frac{4(x-4)}{9(2x-1)} \times \frac{(2x-1)}{(x-4)(x-4)} = \frac{4\cancel{(x-4)}}{9(2x-1)} \times \frac{(2x-1)}{\cancel{(x-4)}(x-4)}$$

$$\frac{4}{9(2x-1)} \times \frac{(2x-1)}{(x-4)} = \frac{4}{9\cancel{(2x-1)}} \times \frac{\cancel{(2x-1)}}{(x-4)} \qquad \boxed{\text{Final Answer } \frac{4}{9(x-4)}}$$

5.2 Rational Expression Division

Dividing rational expressions will build on the same concepts that were discussed in the last section, regarding multiplication of rational expressions. Thinking in terms of components is helpful here. If reciprocals and rational number division are reinforced then ration expression division builds upon these concepts.

Definition 5.2 — Reciprocal. For a real number n, the reciprocal is $\frac{1}{n}$. A reciprocal is also described as the multiplicative inverse or the inverse. The reciprocal of $\frac{a}{b} = \frac{b}{a}$.

To find the reciprocal of a fraction, one simply flips the fraction. This is demonstrated, below.

The reciprocal of $\frac{3}{5}$ is $\frac{5}{3}$ The reciprocal of $\frac{6}{8}$ is $\frac{8}{6}$

The reciprocal of any whole number is 1 divided by that whole number.

The reciprocal of 2 is $\frac{1}{2}$ The reciprocal of 10 is $\frac{1}{10}$

Solved Problem 5.5 Find the reciprocals.

(a) $\frac{3}{7}$ (b) 8 (c) $\frac{5}{8}$ (d) 225

(a) The reciprocal of $\frac{3}{7}$ is $\boxed{\frac{7}{3}}$

(b) The reciprocal of 8 is $\boxed{\frac{1}{8}}$

(c) The reciprocal of $\frac{5}{8}$ is $\boxed{\frac{8}{5}}$

(d) The reciprocal of 225 is $\boxed{\frac{1}{225}}$

The concept of a reciprocal really simplifies the division of rational expressions. This is demonstrated, first, with rational numbers.

$$\frac{5}{8} \div \frac{5}{8} = \frac{5}{8} \times \boxed{\frac{8}{5}} = \frac{\cancel{5}}{\cancel{8}} \times \frac{\cancel{8}}{\cancel{5}} = \frac{1}{1} = \boxed{1} \qquad \frac{3}{7} \div \frac{5}{9} = \frac{3}{7} \times \boxed{\frac{9}{5}} = \frac{3(9)}{7(5)} = \boxed{\frac{27}{35}}$$

One can see, just above, that dividing by a rational number is equal to multiplying by the reciprocal of that rational number. Just above one can see that in order to divide by $\frac{5}{8}$, one multiplies by its reciprocal $\frac{8}{5}$. Just above, one can see that in order to divide by $\frac{5}{9}$, one, again, multiplies by its reciprocal $\frac{9}{5}$.

Solved Problem 5.6 Divide these rational numbers.

(a) $\frac{4}{13} \div \frac{4}{13}$ (b) $\frac{1}{13} \div \frac{3}{2}$ (c) $\frac{2}{7} \div -\frac{1}{3}$

(a) $\frac{4}{13} \div \frac{4}{13} = \frac{4}{13} \times \frac{13}{4} = \frac{\cancel{4}}{\cancel{13}} \times \frac{\cancel{13}}{\cancel{4}} = \frac{1}{1} = \boxed{1}$

(b) $\frac{1}{13} \div \frac{3}{2} = \frac{1}{13} \times \frac{2}{3} = \frac{1(2)}{13(3)} = \boxed{\frac{2}{39}}$

(c) $\frac{2}{7} \div -\frac{1}{3} = \frac{2}{7} \times -\frac{3}{1} = \frac{2(-3)}{7(1)} = \boxed{\frac{-6}{7}}$

Once dividing rational numbers has been reinforced one is ready to divide rational expressions. To divide by a rational expression, again one multiplies by its reciprocal.

$$\frac{5x^2+10x}{5x} \div \frac{3x^2}{3x^2+9x} = \frac{5x^2+10x}{5x} \times \frac{3x^2+9x}{3x^2}$$

In the previous section, the point was made that it is helpful to factor polynomials when multiplying or dividing rational expressions.

$$\frac{5x(x+2)}{5x} \times \frac{3x(x+3)}{3x(x)} = \frac{\cancel{5x}(x+2)}{\cancel{5x}} \times \frac{\cancel{3x}(x+3)}{\cancel{3x}(x)} = \boxed{\frac{(x+2)(x+3)}{x}}$$

Just above, the final rational expression has an x in the variable. Recall that one can not divide by zero, so for this last example, $x \neq 0$, meaning that x can not equal to zero (x is not permissible). Key points to keep in mind while dividing rational expressions:

- Find the reciprocal of the divisor
- Check if factors simplify the division/multiplication operation
- Describe non-permissible values

Solved Problem 5.7 Solve $\frac{x^2+x-20}{x^2-4x-45} \div \frac{x^2-16}{x^2-5x-36}$

$$\frac{x^2+x-20}{x^2-4x-45} \div \frac{x^2-16}{x^2-5x-36} = \frac{x^2+x-20}{x^2-4x-45} \times \boxed{\frac{x^2-5x-36}{x^2-16}}$$

Above, one can see that the first step was to find the reciprocal of the divisor. At this point, one can factor the polynomials.

$$\frac{(x+5)(x-4)}{(x-9)(x+5)} \times \frac{(x-9)(x+4)}{(x+4)(x-4)} = \frac{\cancel{(x+5)}\cancel{(x-4)}}{\cancel{(x-9)}\cancel{(x+5)}} \times \frac{\cancel{(x-9)}\cancel{(x+4)}}{\cancel{(x+4)}\cancel{(x-4)}} = \frac{1}{1} = \boxed{1}$$

This demonstrates how factoring can significantly simplify the problem. ■

Solved Problem 5.8 Solve $\frac{12}{x^2+12x+36} \div \frac{12x}{x^2+6x}$

$$\frac{12}{x^2+12x+36} \div \frac{12x}{x^2+6x} = \frac{12}{x^2+12x+36} \times \boxed{\frac{x^2+6x}{12x}}$$

Above, the first step was to find the reciprocal of the divisor. Then, one can factor the polynomials.

$$\frac{12}{(x+6)(x+6)} \times \frac{x(x+6)}{12x} = \frac{12x(x+6)}{(x+6)(x+6)12x} = \frac{\cancel{12x}\cancel{(x+6)}}{\cancel{(x+6)}(x+6)\cancel{12x}} = \boxed{\frac{1}{x+6}}$$

Notice that if $x=-6$ then $x+6=0$. The denominator can not be zero so x can not equal -6. ■

5.3 Rational Expression Addition

Adding rational expressions requires that the rational expressions have the same denominator. This is reinforced below with rational numbers.

$$\frac{1}{8}+\frac{4}{8}=\frac{1+4}{8}=\boxed{\frac{5}{8}} \qquad \frac{5}{13}+\frac{2}{13}=\frac{5+2}{13}=\boxed{\frac{7}{13}}$$

If rational expressions have different denominators then they will have to be modified, in order to add them. There are two alternatives if the denominators are different.

- Find a common denominator
- Find the lowest common denominator

Finding a common denominator or the lowest common denominator, may seem simple enough, but if a student has not had the chance to practice this then rational expression addition can seem difficult. The real difficulty will be this missing link, of being able to find a common denominator. First, finding a common denominator will be demonstrated and reinforced with rational numbers.

$$\frac{1}{2}+\frac{4}{5} = \frac{1}{2}\left(\frac{5}{5}\right)+\frac{4}{5}\left(\frac{2}{2}\right) = \frac{1(5)}{2(5)}+\frac{4(2)}{5(2)} = \boxed{\frac{5}{10}+\frac{8}{10}} = \frac{5+8}{10} = \boxed{\frac{13}{10}}$$

As shown above, once the two terms have the same denominator, as shown in the orange border, they can be added. Notice above that multiplying the denominators 2 and 5 leads to 10. In this case, simply multiplying the denominators lead to the lowest common denominator. This is not always the case, and sometimes it is unnecessary to multiply the denominators.

Tip Simply multiplying the denominators leads to a common denominator, which is helpful, but it may not be the lowest common denominator

There are two routes that can be used when working with rational expressions with different denominators. The route shown above, makes it possible to add any rational numbers or expressions. Another route is to find the lowest common denominator. The two routes are demonstrated below.

Finding a common denominator:

$$\frac{1}{10}+\frac{4}{100} = \frac{1}{10}\left(\frac{100}{100}\right)+\frac{4}{100}\left(\frac{10}{10}\right) = \frac{1(100)}{10(100)}+\frac{4(10)}{100(10)}$$

$$\boxed{\frac{100}{1000}+\frac{40}{1000}} = \frac{140}{1000} = \frac{14}{100} = \boxed{\frac{7}{50}}$$

Finding the lowest common denominator:

The denominator 100 is already a multiple of 10 so 100 is the lowest common multiple of 10 and 100. So actually it was unnecessary to multiply the denominators, as was done just above. In this case, 100 is the lowest common denominator. This lowest common denominator simplifies the addition process, as shown below.

$$\frac{1}{10}+\frac{4}{100} = \frac{1}{10}\left(\frac{10}{10}\right)+\frac{4}{100} = \boxed{\frac{10}{100}+\frac{4}{100}} = \frac{10+4}{100} = \boxed{\frac{7}{50}}$$

An example is shown below where, again, one denominator is a multiple of the other, and this is demonstrated with rational numbers for simplicity. The same concept applies to the addition of rational expressions.

$$\frac{1}{2}+\frac{1}{8} = \frac{1}{2}\left(\frac{4}{4}\right)+\frac{1}{8} = \frac{1(4)}{2(4)}+\frac{1}{8} = \boxed{\frac{4}{8}+\frac{1}{8}} = \frac{4+1}{8} = \boxed{\frac{5}{8}}$$

Multiples of 2 are 2, 4, 6, $\boxed{8}$, ... One denominator is a multiple of the other. Since the denominator 8 is a multiple of 2, only the other term with the denominator 2 is modified, by multiplying by four. Then, they can be added.

Tip The lowest common denominator is the least common multiple of the denominators

Solved Problem 5.9 Find the lowest common denominator (LCD) for each list of rational expressions.

(a) $\frac{1x}{4}, \frac{3x}{5}$ (b) $\frac{3}{2x}, \frac{5}{4x}$ (c) $\frac{1}{2x^2}, \frac{2}{3x}$ (d) $\frac{1x}{(x+1)(x-1)}, \frac{2}{x(x+1)}$

(a) Multiples of 4 are 4, 8, 16, $\boxed{20}$, 24, ... Multiples of 5 are 5, 10, 15, $\boxed{20}$, 25, ... Observe that 20 is the lowest common multiple of 4 and 5.

$\boxed{\text{Final Answer LCD is 20}}$

(b) Notice that the denominator $4x$ is a multiple of the other denominator $2x$. So $4x$ is the lowest common denominator. This is because $2 \times 2x = 4x$.

$\boxed{\text{Final Answer LCD is } 4x}$

(c) The coefficients 2 and 3 are handled separately from the variables x^2 and x. One needs to find the lowest common multiple of 2 and 3. Multiples of 2 are 2, 4, $\boxed{6}$, ... Multiples of 3 are 3, $\boxed{6}$, 9, 12, ... So, $\boxed{6}$ is the lowest common multiple of 2 and 3. Then x^2 is a multiple of x in the second denominator. So, $\boxed{x^2}$ is the lowest common multiple of x^2 and x. Then 6 and x^2 are combined to arrive at the lowest common denominator.

$\boxed{\text{Final Answer LCD is } 6x^2}$

(d) Multiplying $(x+1)(x-1)$ by $x(x+1)$ would lead to a common denominator, but not the lowest common denominator. Here, each factor in the denominators is treated as a different entity. One of each factor that is present is used to form the lowest common denominator. Both denominators include the factor $(x+1)$. Still, one of these will be part of the lowest common denominator. Then the factor $(x-1)$ and the factor x will be included in the lowest common denominator.

$$\boxed{\text{Final Answer LCD is } (x+1)(x-1)\,x}$$

Confirm that this LCD is a multiple of both denominators.

First denominator $(x+1)(x-1)$ $\quad (x+1)(x-1)\times x = (x+1)(x-1)\,x$

Second denominator $x(x+1)$ $\quad x(x+1)\times(x-1) = (x+1)(x-1)\,x$ ▪

In the last solved problem, the factors of the denominators are fairly clear, or they are provided. Often, it may be necessary to factor the denominators, before finding the lowest common denominator. The following components are helpful with addition of rational expressions.

- Factor the denominators if necessary
- Find the lowest common denominator
- Simplify like terms where possible

If a student is having trouble with adding rational expressions, the real missing link is either factoring or finding the lowest common denominator. Finding the lowest common denominator also involves factoring. Factoring was demonstrated in Section 2.5 Quadratics on page 95. The lowest common denominator is discussed in the solved problem above.

Solved Problem 5.10 Solve $\dfrac{x+1}{5}+\dfrac{3x-2}{3}$

The denominator 5 is not a multiple of the denominator 3, and the denominator 3 is not a multiple of the denominator 5. The next step is to search for the lowest common multiple of 3 and 5.

Multiples of 3 are 3, 6, 9, 12, $\boxed{15}$, ... Multiples of 5 are 5, 10, $\boxed{15}$, 20, ... This means that 15 is the lowest common denominator.

The denominator 5 needs to be multiplied by 3 to equal 15, so the first term is multiplied by $\dfrac{3}{3}$. The second term is divided by $\dfrac{5}{5}$. This makes both denominators the same.

$$\frac{(3)}{(3)}\frac{x+1}{5}+\frac{5}{5}\frac{3x-2}{3} = \frac{3(x+1)}{3(5)}+\frac{5(3x-2)}{5(3)} = \frac{3(x+1)+5(3x-2)}{15}$$

1 2 $\qquad$ 1 2

$$3(x+1) = \boxed{3x+3} \qquad 5(3x-2) = \boxed{15x-10}$$

The numerator is then rewritten and simplified

$$\frac{3x+3+15x-10}{15} = \frac{3x+15x+3-10}{15} \qquad \boxed{\text{Final Answer } \frac{18x-7}{15}}$$

Solved Problem 5.11 Solve $\dfrac{-5x-2}{x^2-2x-3} + \dfrac{2}{x+1}$

The factors of the denominator x^2-2x-3 are $(x-3)(x+1)$. Since this denominator is a multiple of $(x+1)$, it is also the least common denominator.

The factor $(x-3)$ is needed to make the second denominator equal to the lowest common denominator $(x-3)(x+1)$.

$$\frac{-5x-2}{(x-3)(x+1)} + \frac{(x-3)}{(x-3)}\frac{2}{(x+1)} \quad = \quad \frac{-5x-2}{(x-3)(x+1)} + \frac{(x-3)}{(x-3)}\frac{2}{(x+1)}$$

The distributive property is applied to the numerator and denominator of the second term.

1 2

$$2\,(x-3) = \boxed{2x-6}$$

This makes it possible to rewrite the addition expression

$$\frac{\boxed{-5x}-2+\boxed{2x}-6}{(x-3)(x+1)} \qquad \boxed{\text{Final Answer } \frac{-3x-8}{(x-3)(x+1)}}$$

Tip While adding or subtracting rational expressions, if one denominator is a multiple of the other, then it is also the lowest common denominator

5.4 Rational Expression Subtraction

With subtraction of rational expressions one will also keep the following steps in mind.

- Factor the denominators if necessary
- Find the lowest common denominator
- Simplify like terms where possible

There is an algebraic manipulation that is quite helpful when adding or subtracting rational expressions. The binomial $2-x$ is a simple binomial, but it may be necessary to change its order so that it is rewritten as $-1(x-2)$. This can also convert an addition of rational expressions into subtraction of rational expressions. This is demonstrated below.

$$2-x \;=\; \boxed{\text{-1}}(-2)\boxed{-(1)}(x) \;=\; -1(-2+x) \;=\; \boxed{-1(x-2)}$$

One can say that -1 was drawn out or pulled out of the binomial in order to reverse the order and place the variable x first. It can be said that this -1 is in the initial expression, and drawing it out makes it possible to rewrite the expression.

Solved Problem 5.12 Rewrite the following binomials, so, as to place the variable x first.

(a) $5-x$ (b) $122-x$ (c) $4-x$ (d) $9-x$

(a) $5-x = \boxed{-1}(-5)+\boxed{-1}(x) = -1(-5+x) = \boxed{-1(x-5)}$

(b) $122-x = \boxed{-1}(-122)+\boxed{-1}(x) = -1(-122+x) = \boxed{-1(x-122)}$

(c) $4-x = \boxed{-1}(-4)+\boxed{-1}(x) = -1(-4+x) = \boxed{-1(x-4)}$

(d) $9-x = \boxed{-1}(-9)+\boxed{-1}(x) = -1(-9+x) = \boxed{-1(x-9)}$ ■

Solved Problem 5.13 Solve $\dfrac{2}{x-7}+\dfrac{3}{7-x}$

Note that $7-x = -1(-7)-1(x) = -1(-7+x) = -1(x-7)$

Also $\dfrac{1}{-1(x-7)} = \dfrac{-1}{x-7}$ so $\dfrac{3}{7-x} = \dfrac{3}{-1(x-7)} = \boxed{\dfrac{-3}{(x-7)}}$

Now, the initial expression is rewritten as a subtraction.

$$\frac{2}{x-7} - \boxed{\frac{3}{(x-7)}} = \frac{2-3}{x-7} \qquad \boxed{\text{Final Answer } \frac{-1}{x-7}}$$ ■

Solved Problem 5.14 Solve $\dfrac{3}{x^2-4}-\dfrac{1}{2-x}$

Solving this problem brings into play the components of factoring, modifying a binomial, finding the lowest common denominator, and the distributive property. This problem can seem challenging but notice that all 4 components are fairly simple. The factors of x^2-4 are $(x+2)(x-2)$

The second denominator is the binomial that will be modified. Note that $2-x = -1(-2)-1(x) = -1(-2+x) = -1(x-2)$

Also $\dfrac{1}{-1(x-2)} = \dfrac{-1}{x-2}$ so $\dfrac{1}{2-x} = \dfrac{1}{-1(x-2)} = \boxed{\dfrac{-1}{(x-2)}}$

Now, the initial expression can be rewritten. $\dfrac{3}{(x+2)(x-2)} - \boxed{\dfrac{-1}{(x-2)}}$

The first denominator is a direct multiple of the second denominator $(x-2)$, so it is also the least common denominator.

The factor $(x+2)$ is needed to make the second denominator equal to the least common denominator. So this is used to form a rational expression that is equal to 1. This rational expression that is equal to 1 is used to modify the second term.

$$1 = \frac{(x+2)}{(x+2)} \quad \text{Then} \quad \frac{3}{(x+2)(x-2)} - \boxed{\frac{-1}{(x-2)}\frac{(x+2)}{(x+2)}}$$

The distributive property is used to solve $-1(x+2)$

$$-1(x+2) = (-1)x + (-1)(2) \quad = -x-2$$

$$\frac{3}{(x+2)(x-2)} - \boxed{\frac{-x-2}{(x+2)(x-2)}} \quad \text{Then} \quad \frac{3-(-x-2)}{(x+2)(x-2)} = \frac{3-1(-x-2)}{(x+2)(x-2)}$$

The distributive property is used again to solve $-1(-x-2)$

$$-1(-x-2) = (-1)(-x) + (-1)(-2) \quad = \boxed{x+2}$$

$$\frac{3+\boxed{x+2}}{(x+2)(x-2)} \qquad \boxed{\text{Final Answer } \frac{x+5}{(x+2)(x-2)}}$$

5.5 Rational Equations & Functions

A polynomial can be set equal to zero in order to find zeros, roots, or solutions of the polynomial. A first step is to find points where the polynomial touches the x-axis. Likewise, one rational expression can be set equal to another rational expression. This can be done in order to find values of x that satisfy both rational expressions. For some value or values of x, both rational expressions are equal. This may or may not be true. Cross-multiplication makes it possible to search for values of x, where both rational expressions are equal.

$$\frac{5}{x+2} = \frac{8}{4x-2} \quad \text{cross-multiply} \qquad \frac{5}{x+2} \quad \frac{8}{4x-2}$$

This leads to $5(4x-2) = 8(x+2)$

$$5(4x-2) = (5)(4x)+(5)(-2) \quad \text{and} \quad 8(x+2) = (8)(x)+(8)(2)$$

$$5(4x)+5(-2)+8(x)+8(2) \;=\; 20x-10+8x+16 = 20x+8x-10+16 = \boxed{28\text{x}+6}$$

Then $28x = 6$ and $\cancel{28}\dfrac{x}{\cancel{28}} = \dfrac{6}{28}$ becomes $x = \dfrac{6}{28} = \boxed{\dfrac{3}{14} = x}$

Recall, that one cannot divide by zero. So, the solution $x = \dfrac{3}{14}$ must be checked in each denominator of the initial equation to make sure that it does not cause division by zero.

For denominator $x+2$ if $x = \dfrac{3}{14}$ then $\dfrac{3}{14}+2 = \dfrac{3}{14}+\dfrac{28}{14} = \dfrac{31}{14} \neq 0$

For the denominator $4x-2$ if $x = \dfrac{3}{14}$ then $4\dfrac{3}{14}-2 = \dfrac{12}{14}-\dfrac{28}{14} = -\dfrac{16}{14} \neq 0$

Since for both denominators the solution $x = (3/14)$ does not cause division by zero, then it is a valid solution. Sometimes a possible solution does cause division by zero, in the denominator. Such a possible solution would not be a valid solution, and such a solution is called an extraneous solution.

Solved Problem 5.15 Solve $\dfrac{x}{x+1} = \dfrac{3}{x+1}$

$$\frac{x}{x+1} \;\bowtie\; \frac{3}{x+1}$$

This leads to $x\,(x+1) = 3\,(x+1)$

$x\,(x)+x(1) = 3\,(x)+3(1)$ becomes $x^2+x = 3x+3$

The terms $3x$ and 3 are subtracted from both sides to have this equation equal to zero.

$x^2+x-3x-3 = 3x-3x+3-3$ leads to $x^2-2x-3 = 0$

$x^2-2x-3 = 0$ has the factors $(x-3)\,(x+1)$

Possible solutions of $(x-3)\,(x+1) = 0$ are $x = 3$ or $x = -1$

These solutions must be tested to make sure that they are not extraneous solutions. Since the denominator is $x+1$:

If $x = 3$ then $x+1$ is $3+1 = 4$ not zero, so 3 is a valid solution

If $x = -1$ then $x+1$ is $-1+1 = 0$ -1 is not a valid solution

The only valid solution is $\boxed{\text{Final Answer } x = 3}$ ■

Understanding rational functions requires knowing complex fractions. Complex numbers combine real and imaginary parts. Similarly, complex fractions have fractions within them. They're like fractions of fractions. Examples are below. "Complex" means having multiple components, here, whole numbers and fractions. This organization helps grasp the concepts.

- Regular fraction: has whole numbers in the numerator and denominator
- Complex fraction: has fractions in the numerator or denominator. It may also have whole numbers in the numerator or denominator.
- Rational expression: has polynomials or monomials in the numerator and denominator.

Technically x^2, 5, and $9x^3$ are monomials not polynomials. Still, we can refer to monomials as polynomials. A monomial is allowed in a rational expression. Examples of complex fractions are shown below.

$$\frac{4}{\frac{1}{2}+\frac{1}{2}} \qquad \frac{5+\frac{2}{3}}{4} \qquad \frac{x^2+\frac{1}{2}}{5} \qquad \frac{9}{5-\frac{5}{6}}$$

After observing the complex fractions above it can be said that each complex fraction consists of fractions within. For each complex fraction, its components consist of at least one fraction. This doesn't make complex fractions difficult. It is worthwhile to reinforce operations with fractions.

$$\frac{\frac{3}{1}}{\boxed{\frac{1}{2}}} \text{ is equal to } \quad \frac{3}{1}\boxed{\left(\frac{2}{1}\right)} = \frac{3(2)}{1(1)} = \frac{6}{1} = 6$$

In the complex fraction, just above, the reciprocal of $\frac{1}{2}$ is $\frac{2}{1}$ and this is multiplied by $\frac{3}{1}$. One can see that division by a fraction is equal to multiplication by the reciprocal of that fraction.

Solved Problem 5.16 Simplify the following complex fractions

(a) $\frac{10}{\frac{1}{2}}$ (b) $\frac{\frac{2}{4}}{\frac{1}{3}}$ (c) $\frac{4}{\frac{1}{2}+\frac{1}{2}}$ (d) $\frac{\frac{3}{2}+\frac{1}{2}}{\frac{1}{5}}$

(a) $\frac{10}{\boxed{\frac{1}{2}}}$ is equal to $\frac{10}{1}\boxed{\left(\frac{2}{1}\right)} = \frac{10(2)}{1} = \frac{20}{1} = \boxed{20}$

(b) $\dfrac{\frac{2}{4}}{\boxed{\frac{1}{3}}}$ is equal to $\dfrac{2}{4}\boxed{\left(\frac{3}{1}\right)} = \dfrac{2(3)}{4(1)} = \dfrac{6}{4} = \boxed{\dfrac{3}{2}}$

(c) $\dfrac{4}{\frac{1}{2}+\frac{1}{2}}$ is equal to $\dfrac{4}{\boxed{\frac{2}{2}}} = \dfrac{4}{1}\boxed{\left(\frac{2}{2}\right)} = \dfrac{4}{1}\left(\frac{1}{1}\right) = 4(1) = \boxed{4}$

(d) $\dfrac{\frac{3}{2}+\frac{1}{2}}{\frac{1}{5}}$ is equal to $\dfrac{\frac{4}{2}}{\boxed{\frac{1}{5}}} = \dfrac{4}{2}\boxed{\left(\frac{5}{1}\right)} = \dfrac{2}{1}\boxed{\left(\frac{5}{1}\right)} = \dfrac{2(5)}{1} = \boxed{10}$ ■

Having discussed and reinforced complex fractions it is possible to discuss a rational function. The graph of a rational function is shown below.

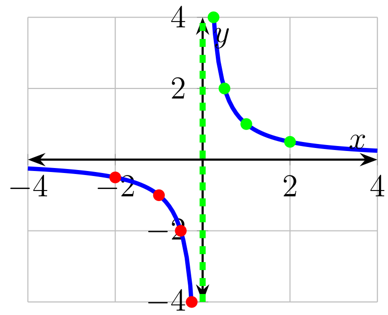
If $x = 2$ then $f(x) = \dfrac{1}{2}$

If $x = 1$ then $f(x) = 1$

If $x = 1/2$ then $f(x) = 2$

If $x = 1/4$ then $f(x) = 4$

Fig. 5.1: $f(x) = \dfrac{1}{x}$



The vertical, green, dashed line in Figure 5.1, just above, is called an asymptote. This is a vertical asymptote. The graph, just above, also has a horizontal asymptote, that is not shown. This will be shown in the next graph.

Definition 5.3 — Asymptote. A line that is approached but never touched.

Notice in the graph, just above, that the blue function approaches the vertical, green, dashed line, but it never touches it. An asymptote can also be horizontal or slanted. Consider below how rational fractions make it possible to understand and observe the shape of the rational function $f(x) = \dfrac{1}{x}$.

If $f(x) = \dfrac{1}{x}$ and $x = 2$ then $f(2) = \dfrac{1}{2}$. This is shown above as the green point $\left(2, \dfrac{1}{2}\right)$.

If $f(x) = \dfrac{1}{x}$ and $x = 1$ then $f(2) = \dfrac{1}{1} = 1$. This is shown above as the green point $(1, 1)$.

If $f(x) = \frac{1}{x}$ and $x = \frac{1}{2}$ then $f\left(\frac{1}{2}\right) = \frac{1}{\frac{1}{2}} = \frac{1}{1}\left(\frac{2}{1}\right) = \frac{1(2)}{1(1)} = \frac{2}{1} = 2$. This is shown above as the green point $\left(\frac{1}{2}, 2\right)$.

If $f(x) = \frac{1}{4}$ and $x = \frac{1}{4}$ then $f\left(\frac{1}{4}\right) = \frac{1}{\frac{1}{4}} = \frac{1}{1}\left(\frac{4}{1}\right) = \frac{1(4)}{1(1)} = \frac{4}{1} = 4$. This is shown above as the green point $\left(\frac{1}{4}, 4\right)$.

The nature of an asymptote can be better discussed with complex fractions. In the graph, above, if the blue function approaches the asymptote from the right side then the value of x decreases, or it would be a value near zero.

For example if $x = \frac{1}{100}$ then $f\left(\frac{1}{100}\right) = \frac{1}{\frac{1}{100}} = \frac{1}{1}\frac{100}{1} = \frac{1(100)}{1(1)} = 100 = y = f(x)$.

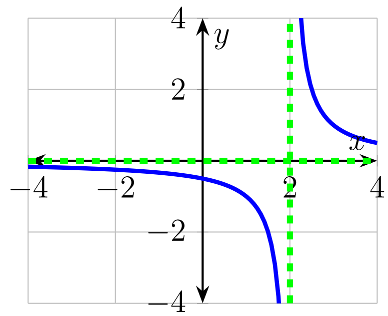
One can see then that as the value of x decreases the value of y increases. One can try to use a lower and lower value of x, but the value of y will only increase greatly, but it will not touch the asymptote.

This graph is the same as the last graph, but it is shifted 2 units to the right.

Again the vertical, dashed, green line shows a vertical asymptote.

The horizontal, dashed, green line shows a horizontal asymptote.

Fig. 5.2: $f(x) = \frac{1}{x-2}$



How would one describe the domain and range of a rational function?

For $f(x) = \frac{1}{x}$ the asymptote occurs at $x = 0$. Conceptually the domain is then all real values other than zero. This same concept is described in three different ways below.

All $x \neq 0$ $\quad$ $(-\infty, 0) \cup (0, \infty)$ $\quad$ $\{x \mid x \in \mathbb{R},\ x \neq 0\}$

The first description of the domain states again that x is equal to all values, other than zero. The second description above is in set-builder notation, which is demonstrated in Solved Problem 2.7. Recall that the symbol $\cup$ refers to the union of two sets. The third description above is in interval notation, which is demonstrated in Example 2.1. Likewise, the range of $f(x) = \dfrac{1}{x}$ would be described in the following way:

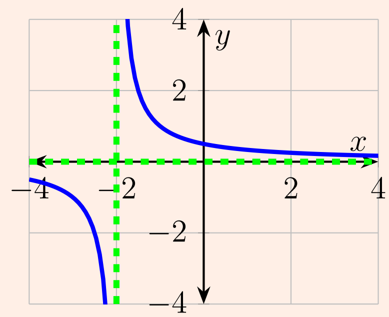
$$\text{All } y \neq 0 \qquad (-\infty, 0) \cup (0, \infty) \qquad \{y \mid y \in \mathbb{R},\ y \neq 0\}$$

Solved Problem 5.17 For the graph below, describe the domain and range.

Final Answer

Domain: All $x \neq -2$
or $(-\infty, -2) \cup (-2, \infty)$
or $\{x \mid x \in \mathbb{R},\ x \neq -2\}$

Range: All $y \neq 0$
or $(-\infty, 0) \cup (0, \infty)$
or $\{y \mid y \in \mathbb{R},\ y \neq 0\}$



5.6 Rational Function Inequalities

Sets and intervals were quite helpful while describing inequalities, and they will be helpful while working with rational inequalities.

$$\frac{x-1}{x+3} \geq -1 \quad \text{Then 1 is added to both sides}$$

$$\frac{x-1}{x+3} + 1 \geq -1 + 1 \quad \text{Leads to} \quad \frac{x-1}{x+3} + 1 \geq 0$$

Addition of fractions requires the same denominator, but $1 = \dfrac{x+3}{x+3}$

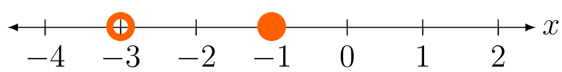
$$\frac{x-1}{x+3} + 1 = \frac{x-1}{x+3} + \frac{x+3}{x+3} = \frac{x-1+x+3}{x+3} = \frac{2x+2}{x+3} \quad \text{Then} \quad \boxed{\frac{2x+2}{x+3} \geq 0}$$

It is known, that for a fraction or rational expression that the denominator can not be equal to zero. Also, if the numerator is equal to zero, then the rational expression is also equal to zero. These two facts lead to critical points. Then the next step is to set the numerator to zero, and to set the denominator equal to zero.

Numerator: $2x + 2 = 0$ then $2x + 2 - 2 = 0 - 2$ and $2x = -2$ $\quad \dfrac{2x}{2} = \dfrac{-2}{2}$ so $\boxed{x = -1}$

Denominator: $x + 3 =$ then $x + 3 - 3 = -3$ and $\boxed{x = -3}$

These two values of x can then be shown on a number line, as shown below. Recall that the denominator can not be equal to zero, and in this case the denominator led to the critical point -3. For this reason, at $x \neq -3$, on the number line below, one can see an open, hollow dot. This is because x can not equal -3. Setting the numerator equal to zero led to the critical point -1, and for this reason at $x = -1$, on the number line below, one can see a closed dot. The value of x can be equal to -1.



$x \neq -3$ and $x = -1$

The two points on the number line, above, leave 3 regions of interest. To finalize the solution one needs to test a point in each region of interest. One tests each region to see if a value in each region satisfies the condition $\dfrac{2x+2}{x+3} \geq 0$. In this case, a value of x in each region should lead to a y value greater than or equal to zero.

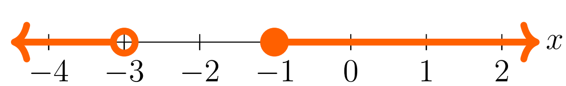
First region: If $x = -4$ then $\dfrac{2x+2}{x+3} = \dfrac{2(-4)+2}{-4+3} = \dfrac{-8+2}{-1} = \dfrac{-6}{-1} = 6 \geq 1$ This is true

For the first region one can see that a value makes the expression $\dfrac{2x+2}{x+3} \geq 0$ true.

Second region: If $x = -2$ then $\dfrac{2x+2}{x+3} = \dfrac{2(-2)+2}{-2+3} = \dfrac{-4+2}{1} = \dfrac{-2}{1} = -2 \geq 1$ Not true

Third region: If $x = 1$ then $\dfrac{2x+2}{x+3} = \dfrac{2(1)+2}{1+3} = \dfrac{2+2}{1} = \dfrac{4}{1} = 4 \geq 1$ This is true

These three tests make it possible to know which regions of the number line are part of the solution. Based on these tests the first region is part of the solution. The second region is not part of the solution. The third region is part of the solution. This solution is shown below.



This solution can be described as $(-\infty, -3) \cup [-1, \infty)$ in set-builder notation, or as $\{x \mid \ x < -3, \ x \geq -1\}$ in interval notation.

5.7 Rational Functions Problems

Click on the page number to the right to see the chapter discussion. Click on the solution number to navigate to the solution.

1. Which expressions below are a ratio of two polynomials? (Page 164) (Solution 1)

(a) $\dfrac{5x}{9\sqrt{5}+15}$ (b) $\dfrac{2x^2+5}{20x^{(-2)}-3}$ (c) $\dfrac{x^2+4}{11x^2-3}$ (d) $\dfrac{3x^4+x^2}{x^2+11}$

2. Which expressions below are a ratio of two polynomials? (Page 164) (Solution 2)

(a) $\dfrac{12x+13}{\sqrt{13}}$ (b) $\dfrac{5+}{20x^{(-2)}-3}$ (c) $\dfrac{2x^{(1/3)}+4}{11x^2}$ (d) $\dfrac{7+x^{(1/2)}}{x^3+1}$

3. Simplify the following rational expressions. (Page 165) (Solution 3)

(a) $\dfrac{x^{10}}{x^9}$ (b) $\dfrac{12x^6}{6x^8}$ (c) $\dfrac{x+3}{x^2+12x+27}$ (d) $\dfrac{(x+2)(x+3)}{x(x-2)(x+2)}$

4. Simplify the following rational expressions. (Page 165) (Solution 4)

(a) $\dfrac{x^4(x+2)}{x^5(x+2)^2}$ (b) $\dfrac{3x^4}{15x^3}$ (c) $\dfrac{(x-4)(x+1)}{8(x-4)}$ (d) $\dfrac{x^2+6x+8}{x(x+4)}$

5. Solve $5x \times \dfrac{x+3}{x^2}$. (Page 166) (Solution 5)

6. Solve $\dfrac{x+5}{x-8} \times \dfrac{x+4}{x+3}$. (Page 167) (Solution 6)

7. Solve $\dfrac{3x^3}{5} \times \dfrac{25}{7x^4}$. (Page 166) (Solution 7)

8. Find the reciprocals of these rational numbers. (Page 167) (Solution 8)

(a) $\dfrac{5}{3}$ (b) $\dfrac{12}{13}$ (c) $\dfrac{1}{7}$ (d) $\dfrac{51}{26}$

9. Find the reciprocals of these rational expressions. (Page 167) (Solution 9)

(a) $\dfrac{x^2}{4}$ (b) $\dfrac{x^3}{2x+11}$ (c) $\dfrac{x^2-5x-24}{1}$ (d) $\dfrac{x^2-4x+7}{x}$

10. Solve $\dfrac{9}{x-1} \div \dfrac{x-5}{1}$. (Page 169) (Solution 10)

11. Solve $\dfrac{x+5}{x-4} \div \dfrac{x+1}{x}$. (Page 169) (Solution 11)

12. Solve $\dfrac{x-3}{x+3} \div \dfrac{x-3}{x+1}$. (Page 169) (Solution 12)

13. Find the lowest common denominator of each pair of rational expressions.

(a) $\dfrac{1}{x+2}, \dfrac{1}{x-1}$ (b) $\dfrac{1}{x^2-2x-3}, \dfrac{1}{x-3}$ (Page 171) (Solution 13)

14. Solve $\dfrac{4x^3}{7} + \dfrac{2x^3}{7}$. (Page 172) (Solution 14)

15. Solve $\dfrac{7}{x-2} + \dfrac{3x}{4x-8}$. (Page 173) (Solution 15)

16. Solve $\dfrac{11x}{x-3} + \dfrac{5x}{3-x}$. (Page 173) (Solution 16)

17. Rewrite the following binomials, so, as to reverse the order of the two terms.

(a) $2-x$ (b) $323-x$ (Page 174) (Solution 17)

18. Solve $\dfrac{19}{3x^2} - \dfrac{17}{3x^2}$.
(Page 174) (Solution 18)

19. Solve $\dfrac{4x^2+7x-10}{3x-2} - \dfrac{4x^2+7x+6}{3x-2}$.
(Page 174) (Solution 19)

20. Solve $\dfrac{2x}{(x+4)(x-4)} - \dfrac{1}{4-x}$.
(Page 174) (Solution 20)

21. Solve $\dfrac{6}{x-5} - \dfrac{x+2}{5-x}$.
(Page 174) (Solution 21)

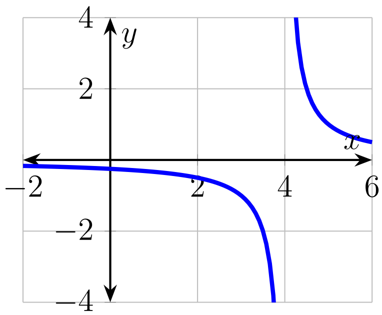
22. Simplify the following complex fractions.
(a) $\dfrac{\frac{1}{5}+\frac{2}{5}}{\frac{3}{5}}$ (b) $\dfrac{\frac{2}{3}+\frac{7}{3}}{2}$ (Page 177) (Solution 22)

23. Simplify the following complex fractions.
(a) $\dfrac{\frac{12}{7}+\frac{16}{7}}{\frac{1}{7}}$ (b) $\dfrac{\frac{5}{4}+\frac{31}{4}}{5}$ (Page 177) (Solution 23)

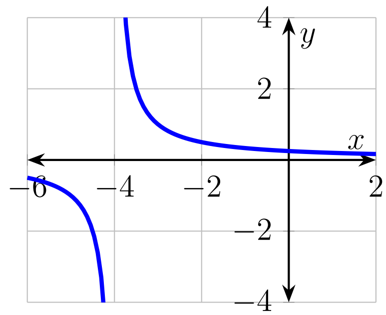
24. Solve $\dfrac{5}{x+1} = \dfrac{10}{x-1}$.
(Page 176) (Solution 24)

25. Solve $\dfrac{2}{x^2-x} = \dfrac{2}{2x-2}$.
(Page 176) (Solution 25)

26. Identify the asymptotes of the function $f(x) = \dfrac{1}{x-4}$ that is graphed below. Describe the domain of this rational function.
(Page 180) (Solution 26)



27. Identify the asymptotes of the function $f(x) = \dfrac{1}{x+4}$ that is graphed below. Describe the domain of this rational function.
(Page 180) (Solution 27)



28. Solve $\dfrac{-x-3}{x+2} \geq 0$.
(Page 180) (Solution 28)

29. Solve $\dfrac{3x-10}{x-4} > 2$.
(Page 180) (Solution 29)

30. Solve $\dfrac{4x}{x-6} < 1$.
(Page 180) (Solution 30)

5.8 Rational Functions Solutions

1. Only (c) and (d)
2. None
3. (a) x (b) $\dfrac{2}{x^2}$ (c) $\dfrac{1}{x+9}$ (d) $\dfrac{x+3}{x(x-2)}$ or $\dfrac{x+3}{x^2-2x}$
4. (a) $\dfrac{1}{x(x+2)}$ or $\dfrac{1}{x^2+2x}$ (b) $\dfrac{x}{5}$ (c) $\dfrac{(x+1)}{8}$ (d) $\dfrac{x+2}{x}$
5. $\dfrac{5x+15}{x}$
6. $\dfrac{x^2+9x+20}{x^2-5x-24}$
7. $\dfrac{15}{7x}$
8. (a) $\dfrac{3}{5}$ (b) $\dfrac{13}{12}$ (c) $\dfrac{7}{1}$ (d) $\dfrac{26}{51}$
9. (a) $\dfrac{4}{x^2}$ (b) $\dfrac{2x+11}{x^3}$ (c) $\dfrac{1}{x^2-5x-24}$ (d) $\dfrac{x}{x^2-4x+7}$
10. $\dfrac{9}{x^2-6x+5}$ or $\dfrac{9}{(x-5)(x-1)}$
11. $\dfrac{x^2+5x}{x^2-3x-4}$
12. $\dfrac{x^2-2x-3}{x-9}$
13. (a) $\dfrac{1}{(x+2)(x-1)}$ (b) $\dfrac{1}{x^2-2x-3}$ or $\dfrac{1}{(x-3)(x+1)}$
14. $\dfrac{6x^3}{7}$
15. $\dfrac{3x+28}{4(x-2)}$
16. $\dfrac{6x}{x-3}$
17. (a) $-1(x-2)$ (b) $-1(x-323)$
18. $\dfrac{2}{3x^2}$
19. $\dfrac{-16}{3x-2}$
20. $\dfrac{3x+4}{(x+4)(x-4)}$
21. $\dfrac{x+8}{x-5}$
22. (a) 1 (b) $\dfrac{3}{2}$
23. (a) 28 (b) $\dfrac{9}{5}$
24. The solution is $x=-3$.
25. The only solution is $x=2$, because $x=1$ is extraneous.
26. The asymptotes are found at $x=4$ and $y=0$. The domain is all $x\neq 4$ or $(-\infty,4)\cup(4,+\infty)$ or $\{x \mid x\in\mathbb{R},\ x\neq 4\}$
27. The asymptotes are found at $x=-4$ and $y=0$. The domain is all $x\neq -4$ or $(-\infty,-4)\cup(-4,+\infty)$ or $\{x \mid x\in\mathbb{R},\ x\neq -4\}$

28. $\left(-\infty, -\dfrac{5}{2}\right]$

29. $(2, +\infty)$

30. $(-\infty, -2)$

Chapter 6: Functions

OVERVIEW

The sections of this chapter are:

6.1 Function Mapping
6.2 Function Recognition
6.3 Transformations
6.4 Operations with Functions
6.5 Composite & Inverse Functions
6.6 Piecewise Functions

Functions, like systems, have inputs and outputs. For example, consider a car's engine: it takes in fuel (input) and produces motion (output). They are fundamental tools akin to languages, crucial for communication. Recognizing functions as systems is more vital than focusing solely on algebraic symbols. Whether linear or curved, functions share the nature of input-output systems.

OBJECTIVES

By the end of the chapter, a student will be able to:

- Discuss fundamental characteristics of functions
- Recognize the graphs of fundamental functions
- Transform fundamental functions in a variety of ways
- Work with inverse and composite functions
- Grasp how one way functions enable transactions in this modern world

6.1 Function Mapping

Function mapping in algebra is like connecting dots. You have an input (like a number) and it corresponds to only one output (another number). It's straightforward, like drawing a line from one point to another. This is called the "vertical line test" because you can't have more than one output for a single input.

Think of a vending machine. You put in a coin, and you get one specific snack in return. If you put in another coin, you get a different snack. Each coin you insert maps to one particular snack. This is similar to function mapping in algebra. For every input (coin), there's only one output (snack). It's like a simple, direct transaction: one input, one output.

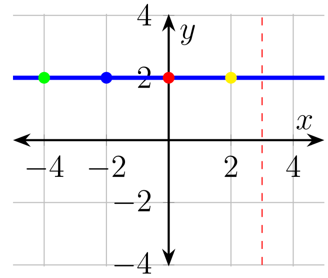
Input $x = -4$ maps to $y = 2$, shown in green

Input $x = -2$ maps to $y = 2$, shown in blue

Input $x = 0$ maps to $y = 2$, shown in red

Input $x = 2$ maps to $y = 2$, shown in yellow

Fig. 6.1: $f(x) = 2$ Map



In Figure 6.1, there's a line that shows how input relates to output. Even though it passes the vertical line test, it's not a one-to-one connection. Instead, it's like having many keys that open the same door. No matter which key you use (input), the door always opens to the same place (output), like always ending up at $y = 2$. The test just checks that each key doesn't open multiple doors, helping us understand how things are connected.

In a function, like the graph above, the independent variable (usually denoted as x) is the input, shown on the horizontal axis. The dependent variable (y) is the output, shown on the vertical axis. It depends on the input value of x, so it's called "dependent."

Below, both a graph and a mapping diagram illustrate many-to-one mapping.

Fig. 6.2: $f(x) = 2$ Map

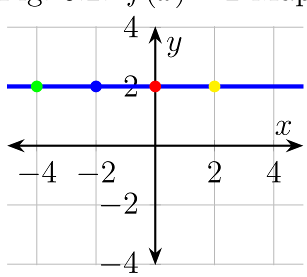
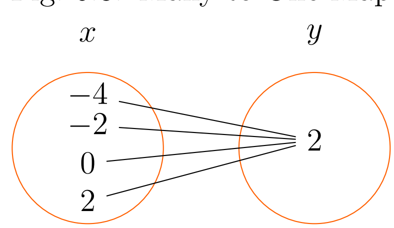


Fig. 6.3: Many-to-One Map



The following observations can be made about the linear function, just above.

- This mapping diagram clearly shows a case of **many-to-one mapping**.
- More than one input leads to the same output.
- Many-to-one mapping is allowed for a linear function.

This is still a linear function. Domain is discussed in Definition 2.8. Range is discussed in Definition 2.9. For the figure, just above, the domain would be $(-\infty, +\infty)$. In set-builder notation the domain would be $\{x \mid x \in \mathbb{R}\}$. And the range would be $y = 2$, because there really is only one possible output value.

For one-to-one mapping, the output would never repeat. Each possible output value would only associate to one possible input. This is demonstrated in the graph below.

Fig. 6.4: $f(x) = x+2$ Map

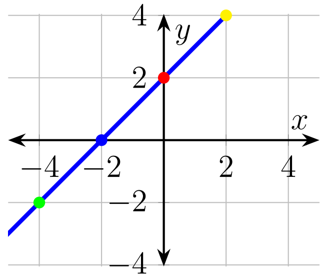
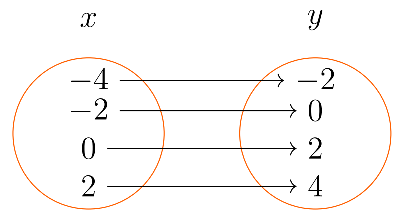


Fig. 6.5: One-to-One Map



The following observations can be made about the linear function, just above.

- This mapping diagram clearly shows a case of **one-to-one mapping**.
- Each input leads to a different output.
- One-to-one mapping is allowed in a linear function.
- Input $x = -4$ maps to $y = -2$, shown in green
- Input $x = -2$ maps to $y = 0$, shown in blue
- Input $x = 0$ maps to $y = 2$, shown in red
- Input $x = 2$ maps to $y = 4$, shown in yellow

For the figure, just above, the domain would be $(-\infty, +\infty)$. In set-builder notation, the domain would be $\{x \mid x \in \mathbb{R}\}$. The range would be $(-\infty, +\infty)$. In set-builder notation, the range would be $\{y \mid y \in \mathbb{R}\}$. The graph, above, illustrates a linear function that displays one-to-one mapping. The mapping diagram below illustrates the concept of one-to-one mapping, in a different way.

A function may have many-to-one mapping, and a function may have one-to-one mapping. What type of mapping cannot occur in a function?

Tip A function will **NOT** display one-to-many mapping

Fig. 6.6: Circle

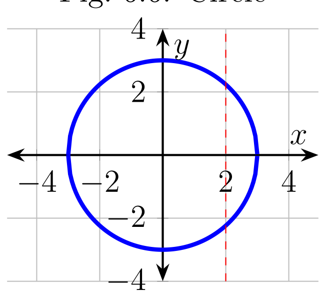
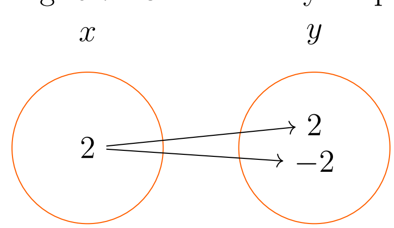
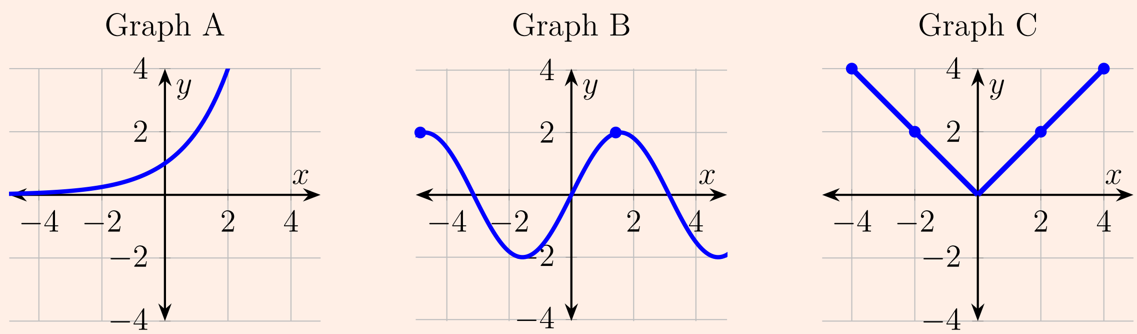


Fig. 6.7: One-to-Many Map



One-to-many mapping is simply the case where one input for x leads to more than one output. For example, just above the input $x = 2$ can map to either $y = +2$ or $y = -2$. The vertical line test is shown with a vertical, dashed, red line. Notice that the vertical line test, greatly simplifies whether one-to-many mapping occurs. If a vertical line test touches a graph more than once, then one-to-many mapping occurs. For a function, a vertical line test, must not touch the graph more than once.

Solved Problem 6.1 Which of the graphs below displays one-to-one mapping?



Notice that for Graph B, an output of $y = 2$ occurs more than once. Graph B shows many-to-one mapping.

Notice that for Graph C, an output of $y = 4$ occurs more than once. Also, an output of $y = 2$ occurs more than once. Graph B shows many-to-one mapping.

For Graph A, each output occurs only once. Any given output value of y never repeats.

Final Answer Only Graph A

The vertical line test was presented in Definition 1.3. Once a function has been confirmed, the horizontal line test can be used to test for one-to-one mapping.

Definition 6.1 — Horizontal Line Test. For a function, if a horizontal line never touches more than one point on the graph, then the function exhibits one-to-one mapping.

In Figure 6.10, the circle isn't a function because it fails the vertical line test, indicating it doesn't have one-to-one mapping. Additionally, it fails the horizontal line test. In Figure 6.9, the parabola is a function but also lacks one-to-one mapping as it fails the horizontal line test; a red dashed line touches it at two points simultaneously. However, in Figure 6.8, the cubic function passes both the vertical and horizontal line tests, making it the only one among them to exhibit one-to-one mapping.

Fig. 6.8: Cubic Function

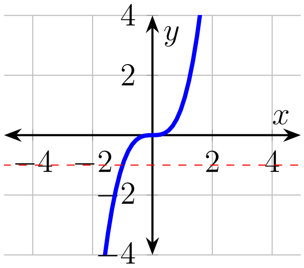


Fig. 6.9: Parabola

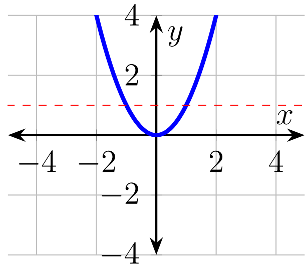
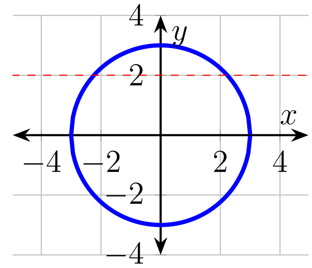


Fig. 6.10: Circle



Tip The vertical line test is used to test if a graph is a function

Tip The horizontal line test is used to test if a function has one-to-one mapping

6.2 Function Recognition

Many concepts have been discussed related to functions. It is worthwhile to organize these concepts in a useful way.

- Exponents
- Linearity
- Nonlinearity
- Domain
- Range
- Asymptotes
- Slope
- Roots or zeros
- Growth or decay

These concepts, above, will be used to describe the six functions below. Figure 6.11 and Figure 6.12, below, show two fundamental graphs that were introduced, in the early chapters of this book. With or without the graph one should be able to see that $f(x) = x$ will be a linear function that shows up as a straight line. It does not have an exponent, so it will not exhibit a curve. This function has a slope that is equal to zero. This is shown in Figure 6.11.

Figure 6.12, shows the graph of $f(x) = x^2$, which is a nonlinear function. The exponent of 2 in this function leads to the parabola or parabolic shape, that is shown.

Fig. 6.11: $f(x) = x$

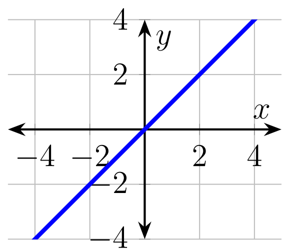
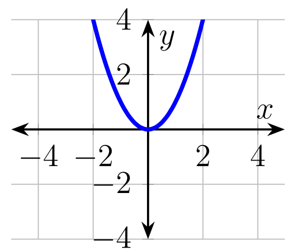


Fig. 6.12: $f(x) = x^2$

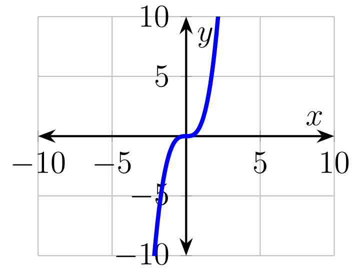


A parabola was discussed and demonstrated on page 81. The function $f(x) = x^2$ has a domain that is all real numbers, and a range of greater than or equal to zero. One can also see its axis of symmetry at the y-axis. It is an even function. Even functions were defined in Definition 4.8.

The domain of $f(x) = x^3$ is all real numbers, and its range is all real numbers. It is an odd function. Odd functions were defined in Definition 4.9. The test or condition for an odd function is that $f(-x) = -f(x)$.

Figure 6.13, shows the graph of $f(x) = x^3$ which is a nonlinear function. Its exponent of 3, tells that it will have curvy behavior. It crosses the origin. This is confirmed by testing an input of zero, as in $f(0) = 0^3 = 0$. So, for an input of zero, the output is zero.

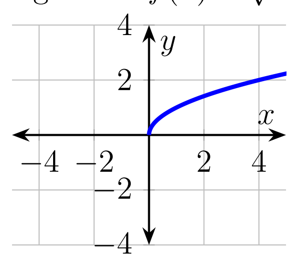
Fig. 6.13: $f(x) = x^3$



The domain of $f(x) = \sqrt{x}$ is all real numbers, that are greater than or equal to zero. The range is also greater than or equal to zero. It is neither an odd function, nor an even function.

Figure 6.14, shows the graph of the square root function $f(x) = \sqrt{x}$. This would be the positive square root function, or the principal square root function.

Fig. 6.14: $f(x) = \sqrt{x}$



One can see that this function, just above, is a nonlinear function. In other words, it is not a straight line. Recall that, for example, the square root of 4 is equal to both positive and negative 2, as in $\sqrt{4} = \pm 2$. Also, recall, that in a function a vertical line test must touch only one point at time. For this reason, the function above includes only the positive solutions. Likewise, the square root function, below, includes the negative solutions.

Figure 6.15, shows the graph of the square root function $f(x) = -\sqrt{x}$. This would be called the negative square root function.

Fig. 6.15: $f(x) = \sqrt{x}$

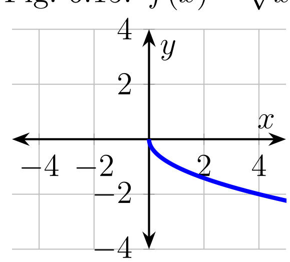
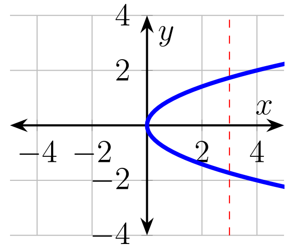


Figure 6.16, shows the graph of $f(x) = \pm\sqrt{x}$. As confirmed by the vertical line test, this is no longer a function. The vertical line test is shown with a red, dashed line.

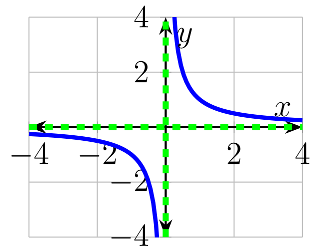
Fig. 6.16: $f(x) = \sqrt{x}$



The domain of $f(x) = \dfrac{1}{x}$ is all real numbers less than zero, and all real numbers greater than zero. Likewise, the range is all real numbers less than zero, and all real numbers greater than zero.

Figure 6.17, shows $f(x) = \dfrac{1}{x}$.
This rational function has two asymptotes. The vertical and horizontal, green, dashed lines show the vertical and horizontal asymptotes.

Fig. 6.17: $f(x) = 1/x$

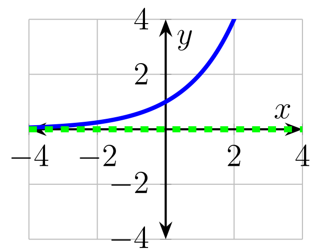


One can see in the graph of $f(x) = \dfrac{1}{x}$ that there are two nonlinear components.

The domain of $f(x) = 2^x$ is all real numbers. The range is all real numbers greater than zero. In other words, the output is always positive as shown by the graph. One can see that this graph is nonlinear, or simply, that it is not a straight line.

Figure 6.18, shows the exponential function $f(x) = 2^x$. The horizontal, green, dashed line is a horizontal asymptote.

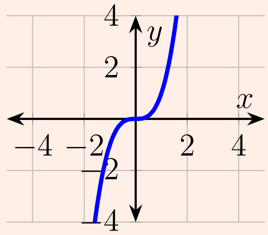
Fig. 6.18: $f(x) = 2^x$



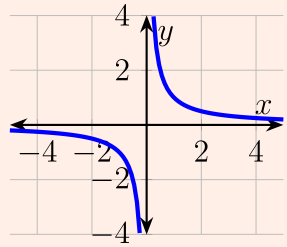
The meaning of the horizontal asymptote in the graph, just above, is that the output y can approach zero, and be close to zero, but the output or y is never zero. At this point, certain basic graphs should associate to their corresponding functions.

Solved Problem 6.2 Identify the function that corresponds to each graph, below.

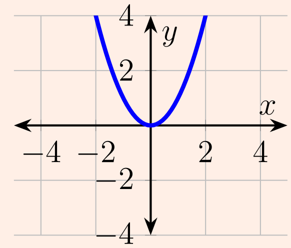
Graph A



Graph B



Graph C



Graph A corresponds to $f(x) = x^3$ Graph B corresponds to $f(x) = \dfrac{1}{x}$

Graph C corresponds to $f(x) = x^2$

6.3 Transformations

The functions reinforced in the previous section can be used to form an endless variety of functions. The transformation of a function refers to:

- shifting to the left
- shifting to the right
- shifting up
- shifting down
- stretching
- compressing
- flipping upside down
- rotation

Slope-intercept form was defined in Definition 1.8. It was explained that the constant in point-slope form describes the y-intercept. A nonzero constant also shifts the function up or down. Below, one can see a basic, linear function with a slope of zero. Then, in Figure 6.18, one can see that adding 2 to the function causes it to shift up.

Fig. 6.19: $y = 0$

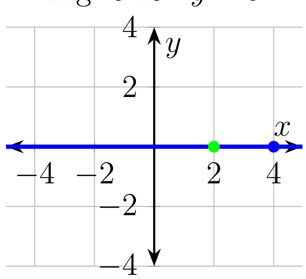


Fig. 6.20: $y = 2$

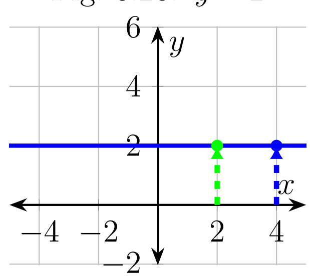
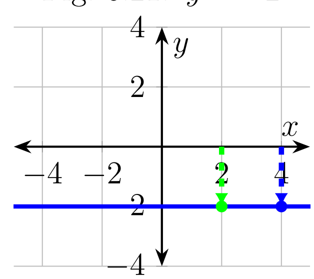
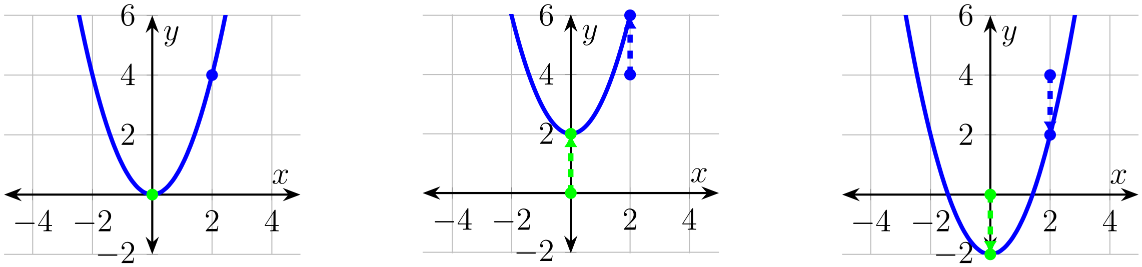


Fig. 6.21: $y = -2$



In Figure 6.19, above, one can see that subtracting 2 causes the function to shift down, two units. This is demonstrated, again, below, with the nonlinear function, $f(x) = x^2$. One can see, in Figure 6.21 that adding 2 to the function causes it to shift up two units. Then in Figure 6.22, on can see that subtracting 2, causes the function to shift down.

Fig. 6.22: $y = x^2$ Fig. 6.23: $y = x^2 + 2$ Fig. 6.24: $y = x^2 - 2$



It is key to remember that if a term is added to a function that adds a constant, then the function will shift up. If a term is added to a function that subtracts a constant, then the function will shift down. A vertical translation, up or down, is an example of a change that is made to the whole function. This is because a constant term is added or subtracted from the whole function.

While working with transformations it is helpful to note that usually one or both of the following occurs.

- A change is made to the whole function
- A change is made to the variable in the function

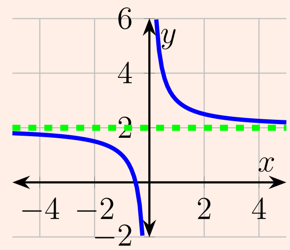
Solved Problem 6.3 Graph the function $f(x) = \dfrac{1}{x} + 2$.

The function $f(x) = \dfrac{1}{x}$ was discussed and graphed in the last section.

The term $\boxed{+2}$ simply causes the function to shift up two units.

One can see that the horizontal asymptote has shifted up two units.

Graph B



A vertical shift up or down is also called translation. Vertical translation is the simplest function transformation. What if one wanted to shift a function to the right or to the left. Shifting to the right or to the left is also called translation. Translation two units to the right is caused by modifying the x variable itself. The variable x is replaced by $(x-2)$. Based on the fundamental order of operations, this often requires the use of parentheses. Notice that 2 is **subtracted** from x.

Tip For translation to the right, x is replaced by $(x - c)$, where c is a constant. Subtraction of a constant associates to translation to the right.

Tip For translation to the left, x is replaced by $(x + c)$, where c is a constant. Addition of a constant associates to translation to the left.

Horizontal translation is demonstrated below. For the three graphs, below, the vertical, green, dashed line shows the axis of symmetry. One can clearly see, in the graph, in Figure 6.24, that the entire parabola has shifted, two units to the left. In Figure 6.25, one can see, that the entire parabola has shifted two units, to the right.

Fig. 6.25: $y = x^2$

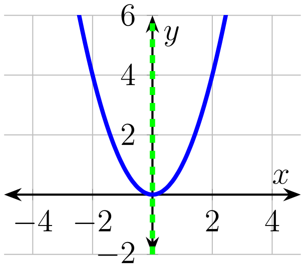


Fig. 6.26: $y = (x+2)^2$

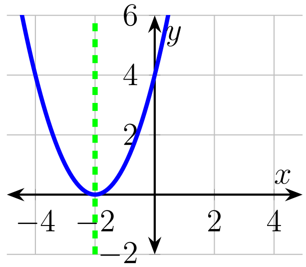
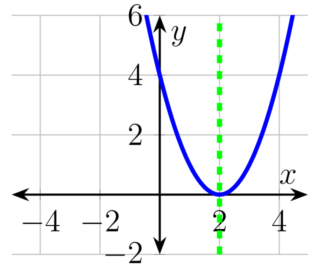


Fig. 6.27: $y = (x-2)^2$

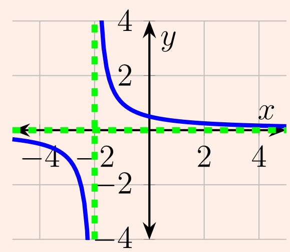


Solved Problem 6.4 Graph the function $f(x) = \dfrac{1}{x+2}$.

The function $f(x) = \dfrac{1}{x}$ was discussed and graphed in the last section.

Since x was replaced by $x+2$, the function shifts to the left two units.

One can see that the horizontal asymptote has not shifted, but the vertical asymptote did shift.



A translation, to the right or the left, is an example of a change that is made only to the variable, within the function. Yes, the function changes as a whole, but notice that the main change is made to the variable.

To reflect a function about the y-axis, imagine the y-axis as a mirror. Whatever is on the right side gets mirrored to the left, and vice versa. So, to reflect a function about the y-axis, you just change the sign of the x-values. In other words, if you have a function $f(x)$, reflecting it about the y-axis is described by $f(-x)$. So, x becomes x.

Reflection about the x-axis is another example of a change that is made to the whole function. For a function $f(x)$, reflection about the x-axis is described by $-f(x)$.

Fig. 6.28: $f(x) = x^2$

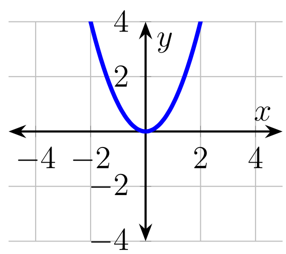
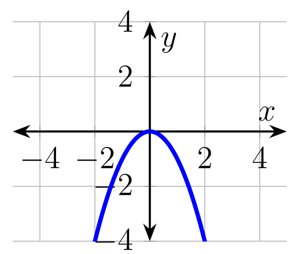


Fig. 6.29: $f(x) = -x^2$



In order to reflect a function about the y-axis, the change is made to the variable, within the function. For a function $f(x)$, reflection about the x-axis is described by $f(-x)$. This is demonstrated below.

In Figure 6.28 and Figure 6.29, the coordinate pair (0, 1) is emphasized with a green point. This green point remains unchanged before and after the reflection about the y-axis.

Fig. 6.30: $f(x) = 2^x$

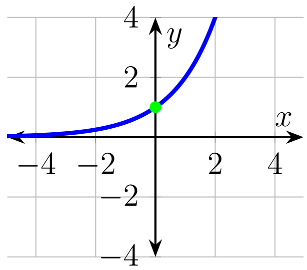
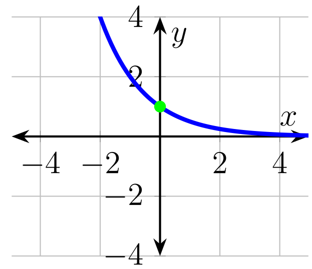


Fig. 6.31: $f(x) = 2^{(-x)}$



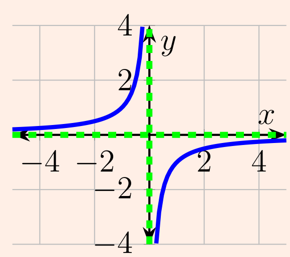
It is worthwhile to emphasize that if the negative is applied to the whole function, then the function $f(x) = 2^x$ becomes $f(x) = -(2^x)$, but this would lead to reflection about the x-axis. For the case of the two graphs, just above, the negative is only applied to the variable within.

Solved Problem 6.5 Graph the function $f(x) = \dfrac{1}{-x}$.

The function $f(x) = \dfrac{1}{x}$ was discussed and graphed in the last section.

Since x was replaced by $-x$, the function reflects about the y-axis.

One can see that the asymptotes do not shift.



What can be expected if a whole function is multiplied by a factor? This is called vertical scaling, and this can cause a function to either stretch or compress. For a function $f(x)$, vertical scaling is described by $f(x) = c(2^x)$. Parentheses are used in this last function to reinforce that the constant factor c is applied to the whole function. This last function could have been written $f(x) = c2^x$, since the exponent is applied before multiplication by c.

Fig. 6.32: $y = 2^x$

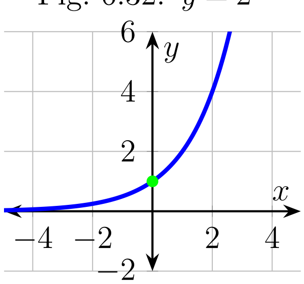


Fig. 6.33: $y = 2(2^x)$

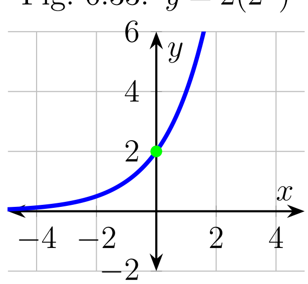
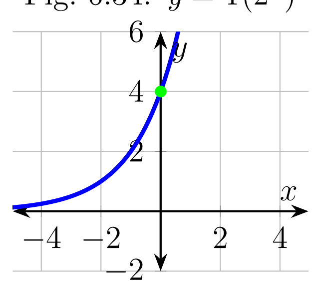


Fig. 6.34: $y = 4(2^x)$



Notice in these three graphs, just above, that as the constant increases, the function is stretched vertically. Notice that the green point at $x=0$, has a y value that increases. For the case of the three graphs above, the constant c is a nonzero whole number, not a fraction. If the constant c is a fraction then the function will compress, as shown in the three graphs below, for the function $f(x)$.

Fig. 6.35: $y = x^2$

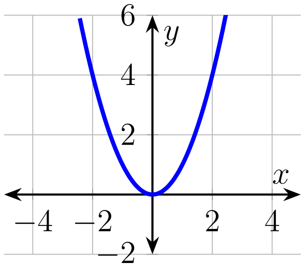


Fig. 6.36: $y = \frac{1}{2}(x^2)$

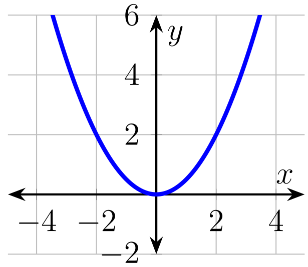
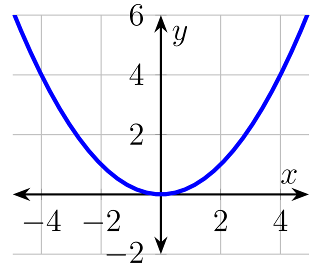


Fig. 6.37: $y = \frac{1}{4}(x^2)$



As one can see, vertical scaling is an example of applying a change to the whole function. For the function $f(x)$, vertical scaling is described as $c \times f(x)$ or $cf(x)$. Scaling can also occur in a horizontal manner.

A scaling factor can also be applied to the variable within, the function, as opposed to being applied to the whole function. This will cause horizontal stretching or compression. For the case of horizontal scaling, a nonzero, whole number factor will cause horizontal compression. For the function $f(x)$, horizontal compression is described by $f(c \times x)$ or $f(cx)$. This is demonstrated below.

Fig. 6.38: $y = x^2$

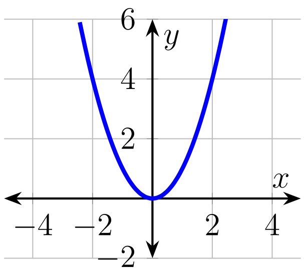


Fig. 6.39: $y = (2x)^2$

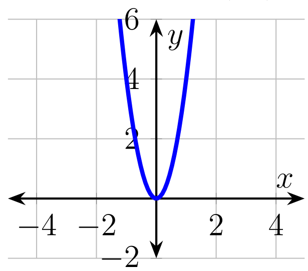
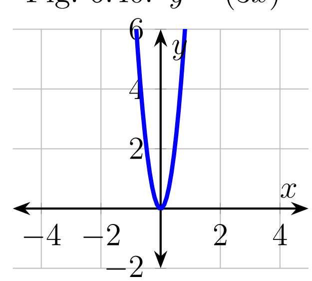


Fig. 6.40: $y = (3x)^2$



For the case of horizontal scaling, a fractional factor will cause horizontal stretching. For the function $f(x)$, horizontal stretching is described by $f\left(\frac{1}{c} \times x\right)$ or $f\left(\frac{1}{c}x\right)$. This is demonstrated below.

Fig. 6.41: $y = x^2$

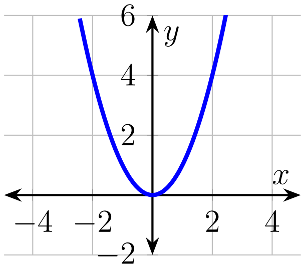


Fig. 6.42: $y = \left(\frac{1}{2}x\right)^2$

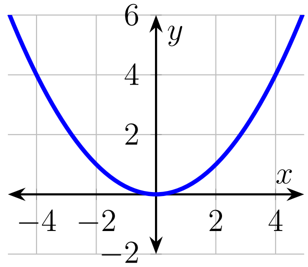
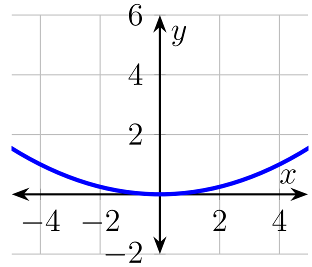


Fig. 6.43: $y = \left(\frac{1}{4}x\right)^2$



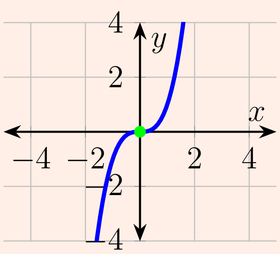
This section has demonstrated several forms of function transformation. A function might involve multiple transformations.

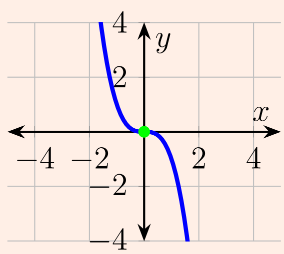
Table 6.1: Function Transformations

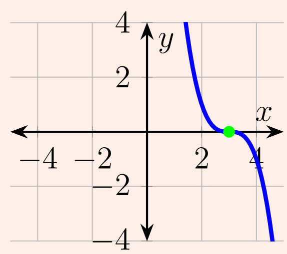
Vertical translation	$f(x)+c$
Horizontal translation	$f(x+c)$
Reflection about x-axis	$-f(x)$
Reflection about y-axis	$f(-x)$
Vertical stretching	$cf(x)$
Vertical compression	$\frac{1}{c}f(x)$
Horizontal stretching	$f\left(\frac{1}{c}x\right)$
Horizontal compression	$f(cx)$

Tip Horizontal stretching/compression, can seem counterintuitive, a nonzero, whole number, factor causes horizontal compression, and a fractional factor causes horizontal stretching

Solved Problem 6.6 Graph the function $f(x) = -(x-3)^3$.







The initial function $f(x)=x^3$ is shown at the top left. The function $f(x)=-(x-3)^3$ consists of two transformations.

The function $f(x)=-\left(x^3\right)$ is shown in the second graph, just above.

In $f(x)=\boxed{-}\left(x^3\right)$ the negative component $\boxed{-}$ leads to a reflection about the x-axis.

This means, that the function is flipped upside-down.

In $f(x)=-\left(x\ \boxed{-3}\right)^3$ the subtraction $\boxed{-3}$ leads to a horizontal translation or shift, to the right. ■

6.4 Operations with Functions

Functions can be added, subtracted, multiplied, and divided. This can be described in the following ways.

- $(f+g)(x)$
- $(f-g)(x)$
- $(f\times g)(x)$
- $\left(\dfrac{f}{g}\right)(x)$

A function is often described with $f(x)$, but a function can also be called or described with $g(x)$, or $h(x)$.

Once these expressions above are understood, operations with functions amount to addition, subtraction, multiplication, and division of polynomials. Polynomial addition was demonstrated in Solved Problem 4.8. Polynomial subtraction was demonstrated in Solved Problem 4.10.

For the case of function addition, $(f+g)(x)=f(x)+g(x)$. For the case of function subtraction $(f-g)(x)=f(x)-g(x)$.

Solved Problem 6.7 $f(x)=4x+1$ and $g(x)=x+2$

Solve (a) $(f+g)(x)$ (b) $(f-g)(x)$

$$(f+g)(x)=4x+1+x+2=4x+x+1+2=\boxed{5x+3}$$

$$(f-g)(x)=4x+1-(x+2)=4x+1-x-2=4x-x+1-2=\boxed{3x-1}$$ ■

For the case of function multiplication, $f(x)\times g(x)$ can be written as $(f\times g)(x)$, which can also be written as $(f\cdot g)(x)$. Polynomial multiplication was demonstrated in Solved Problem 4.12.

Solved Problem 6.8 $f(x) = 5x$ and $g(x) = 4x - 1$ Solve $(f \cdot g)(x)$

$$(f \cdot g)(x) = 5x(4x - 1) = 5x(4x) - 5x(1) = \boxed{20x^2 - 5x}$$

$(f \cdot g)(x)$ is understood as $(f \cdot g)$ as a function of x.

Solved Problem 6.9 $f(x) = 2x - 1$ and $g(x) = 2x - 2$ Solve $(f \cdot g)(x)$

$$(f \cdot g)(x) = (2x - 1)(2x - 2) = 2x(2x - 2) \; -1(2x - 2)$$

$$2x(2x) - 2x(2) - 1(2x) - 1(-2) = 4x^2 - 4x - 2x + 2 = \boxed{4x^2 - 6x + 2}$$

$(f \cdot g)(x)$ is understood as $(f \cdot g)$ as a function of x.

One can see, in the last solved problem, that, multiplication with functions amounts to multiplication of polynomials. Likewise, division of functions is like division of polynomials.

For the case of function division $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$.

Solved Problem 6.10 $f(x) = 10x$ and $g(x) = 8x$ and $h(x) = 2x$

Solve $\left(\frac{g}{h}\right)(x)$

$$\left(\frac{g}{h}\right)(x) = \frac{8x}{2x} = \frac{4(\cancel{2x})}{\cancel{2x}} \qquad \left(\frac{g}{h}\right)(x) = \boxed{4}$$

$\left(\frac{g}{h}\right)(x)$ is understood as $\left(\frac{g}{h}\right)$ as a function of x.

For function division and polynomial division, the denominator can not be a zero value. For function division, it may be necessary to make use of long division or synthetic division. Long division was demonstrated in Solved Problem 4.14. Synthetic division was demonstrated in Solved Problem 4.17.

The expression $\left(\frac{f}{g}\right)(x)$ is the same as $f(x) \div g(x)$ or $\frac{f(x)}{g(x)}$

Long division can now be carried out.

$\left(\frac{f}{g}\right)(x) = \frac{5x^2 - 3x - 36}{x - 3}$ is rewritten

$$x-3\overline{\big)5x^2-3x-36}$$

$\frac{\boxed{5x^2}}{x} = 5x$ leads to the first term in the solution. Multiplying $5x$ by the divisor $x-3$ leads to $5x^2 - 15x$. When this binomial is subtracted it becomes $\boxed{-5x^2 + 15x}$. Subtraction and bringing down the next term in the dividend leads to $\boxed{12x - 36x}$. Next the steps repeat.

$$\begin{array}{r} 5x \\ x-3\overline{\big)\boxed{5x^2}-3x-36} \\ \underline{-5x^2+15x} \\ \boxed{12x-36} \end{array}$$

$\frac{\boxed{12x}}{x} = 12$ leads to the second term in the solution 12. Multiplying 12 by the divisor $x-3$ leads to $12x - 36$. When this binomial is subtracted it becomes $\boxed{-12x + 36}$. Subtraction, and bringing down the next term in the dividend, leads to $\boxed{0}$.

$$\begin{array}{r} 5x+12 \\ x-3\overline{\big)5x^2-3x-36} \\ \underline{-5x^2+15x} \\ \boxed{12x}-36 \\ \underline{-12x+36} \\ \boxed{0} \end{array}$$

The final answer would be $\boxed{\left(\frac{f}{g}\right)(x) = 5x + 12}$ with a remainder of zero.

Solved Problem 6.11 $g(x) = x - 1$ and $h(x) = x^2 - 1$

Solve $\left(\frac{h}{g}\right)(5)$

$\left(\frac{h}{g}\right)(x) = \frac{x^2-1}{x-1} = \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}$ $\left(\frac{h}{g}\right)(x) = x + 1$ $\left(\frac{h}{g}\right)(5) = 5 + 1 = \boxed{6}$

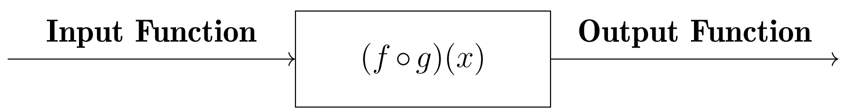
$\left(\frac{h}{g}\right)(5)$ is understood as $\left(\frac{h}{g}\right)$ as a function of 5. Recall that a function has an input and output nature. In this case, 5 is the input, and 6 is the output. ■

In the last solved problem, factoring the numerator made it possible to simplify the division operation.

6.5 Composite and Inverse Functions

It has been demonstrated in different ways that an input value of x leads to an output value of y. What if the input to a function is another function? This would be a composite function. This would be a function of a function, and this is described as $(f \circ g)(x)$. The meaning of this last expression is the function $f(x)$ as a function of $g(x)$, where the function $g(x)$ is input, to the function $f(x)$. This is illustrated in the figure below.

Fig. 6.44: Composite Function Input, Output



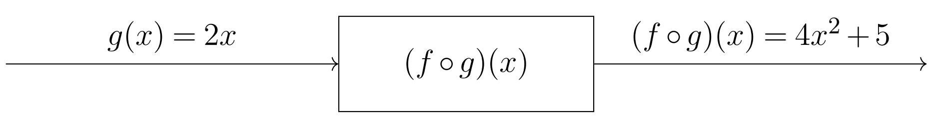
To demonstrate this let $f(x) = x^2 + 5$ and let $g(x) = 2x$. Then for $(f \circ g)(x)$, the function $f(x)$ is the main function, and the input will be $g(x)$.

$$f(x) = x^2 + 5 \text{ then } (f \circ g)(x) = (g(x)) = f(2x) = \boxed{(2x)}^2 + 5$$

$$= (2x)^2 + 5 = 2x(2x) + 5 = \boxed{4x^x + 5 = (f \circ g)(x)}$$

The input, output nature of this composite function is visualized, below.

Fig. 6.45: Composite Function $f(x) = x^2 + 5$, $g(x) = 2x$



Solved Problem 6.12 $f(x) = x^2 + 4$ and $g(x) = 2x - 1$

Solve (a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$

(a) The function $g(x)$ is input into $f(x)$

$$(f \circ g)(x) = f(g(x)) \text{ then } f(2x-1) = (2x-1)^2 + 4 = (2x-1)(2x-1) + 4$$

$$(f \circ g)(x) = (2x-1)(2x-1) + 4 = 2x(2x-1) \quad -1(2x-1) + 4$$

$$2x(2x)+2x(-1)-1(2x)-1(-1)+4 \quad = \quad 4x^2-2x-2x+1+4 \quad = \quad 4x^2-4x+5$$

$\boxed{\text{(a) Final answer } 4x^2-4x+5}$

(b) The function $f(x)$ is input into $g(x)$

$(g \circ f)(x) = g(f(x))$ then $g(x^2+4) = 2\left(x^2+4\right)+1$

1 2

$$(g \circ f)(x) \quad = \quad 2(x^2+4)+1 \quad = \quad 2(x^2)+2(4)+1 \quad = \quad 2x^2+8+1$$

$\boxed{\text{(b) Final answer } 2x^2+9}$

Now, that the input, output nature of composite functions has been emphasized, one may want to reverse the input and output. In other words, if one knows the output, one may want to find the input. An inverse function makes it possible to find the input, if one knows the output. At first, it may be tricky to see why this would be useful, and it may be challenging to understand applications of inverse functions.

If $x=-4$, then $y=-8$

If $x=-2$, then $y=-4$

If $x=0$, then $y=0$

If $x=2$, then $y=4$

Fig. 6.46: $f(x)=2x$ Map

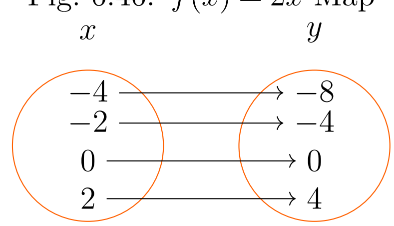
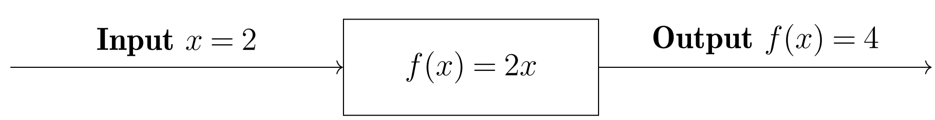


Fig. 6.47: $f(x)=2x$



This example shown in Figure 6.44 and Figure 6.45, is a simple case where the function doubles the input.

Understanding inverse functions is crucial for secure communications and transactions, like online banking and sales. Cryptography relies on making functions hard to invert, ensuring scrambled messages stay secure. Without this, passwords could be easily unscrambled. This chapter focuses on easily invertible functions but highlights the importance of grasping inverse functions for security.

The inverse of a function $f(x)$ is written as $f^{-1}(x)$. How is the inverse of $f(x)=2x$ found algebraically?

If $f(x) = 2x$ then interchange x and y. In other words, in $y = 2x$ replace y with x and replace x with y. The function $y = 2x$ becomes $x = 2y$. Now solve for y, which leads $y = \dfrac{x}{2}$. This means that the inverse of the function $f(x) = 2x$ is described by $f^{-1} = \dfrac{x}{2}$.

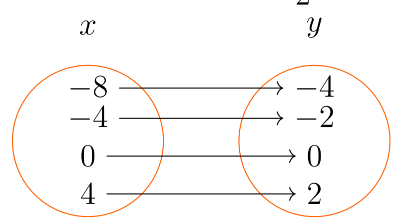
If $x = -8$, then $y = -4$

If $x = -4$, then $y = -2$

If $x = 0$, then $y = 0$

If $x = 4$, then $y = 2$

Fig. 6.48: $f^{-1}(x) = \dfrac{x}{2}$ Map



In Figure 6.46, above, one can see that the inputs are the outputs from Figure 6.44. Each output can now be used to find the original input, in the original function described by $f(x)$. As shown below, the inverse of a function reverses the outputs and inputs.

Fig. 6.49: If $f(x) = 2x$, Then $f^{-1}(x) = \dfrac{x}{2}$

Input $x = 4$ $\longrightarrow$ $\boxed{f(x) = \dfrac{x}{2}}$ $\longrightarrow$ **Output** $f(x) = 2$

The following steps are useful while learning to invert a function.

- Interchange x and y
- Solve for y

The concept of an inverse function is quite simple. One is reversing a function's behavior. In practice, it can make use of some algebraic manipulations. Students often need practice with these algebraic manipulations.

One might see that the reverse of doubling would be to divide in half. The process of inverting a function makes it easier to see this. With more interesting functions the process of inverting the function is necessary.

Solved Problem 6.13 Find the inverse $f^{-1}(x)$ of the function $f(x) = 4x + 3$.

The equation $f(x) = 4x + 3$ is the same as the equation $y = 4x + 3$. Recall that y is the same as the output $f(x)$.

Interchange x and y in $y = 4x + 3$. The equation is converted to $y = 4x + 3$ then leads to $x = 4y + 3$. The next step is to solve for y. First, 3 is subtracted from both sides to begin to isolate y.

$$x \boxed{-3} = 4y + 3 \boxed{-3} \qquad x - 3 = 4y$$

Next, both sides are divided by 4, to further isolate y.

$$\frac{1}{\boxed{4}}(x-3) = \frac{1}{\boxed{4}}4y \qquad \frac{x-3}{4} = \frac{\cancel{4}y}{\cancel{4}} \qquad \boxed{\text{Final Answer } f^{-1}(x) = \frac{x-3}{4}}$$

6.6 Piecewise Functions

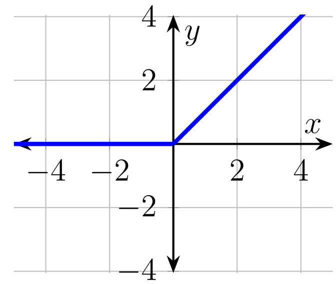
A piecewise function is literally a matter of components. It may look different, but it involves concepts that have been discussed and demonstrated. It is called piecewise because it consists of components, pieces, or sections. Each section will behave differently.

$$f(x) = \begin{cases} 0 & x < 0 \\ x & x \geq 0 \end{cases}$$

A portion of this graph displays a flat, straight line

A portion of this graph displays an incline.

Fig. 6.50: Piecewise Function



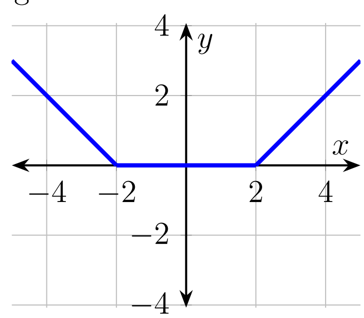
This piecewise function is common in artificial intelligence and neural networks. It's straightforward when you break it into its components. Each component has its own domain. This function has two pieces: one for $x < 0$ and one for $x \geq 0$. If $x < 0$, the output is zero; if $x \geq 0$, the output is x. We've seen linear functions with slopes of zero and one before, but here, different components are combined to form a piecewise function.

$$f(x) = \begin{cases} -x-2 & x < -2 \\ 0 & -2 \leq x \leq 2 \\ x-2 & 2 < x \end{cases}$$

This piecewise function has 3 rows, and it has 3 components or pieces.

All 3 components are linear.

Fig. 6.51: Piecewise Function



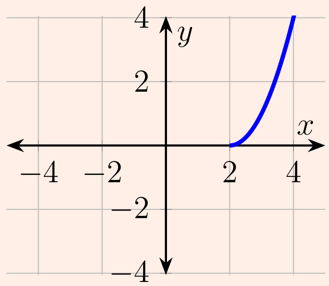
Solved Problem 6.14

Graph the piecewise function $f(x) = \begin{cases} (x-2)^2 & 2 \leq x \\ 0 & -2 \leq x \leq 2 \\ -4x-8 & x \leq 2 \end{cases}$

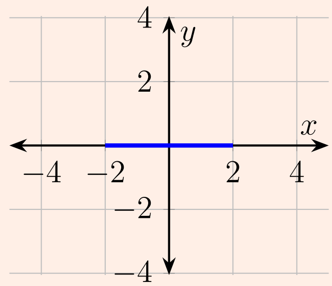
This piecewise function has 3 rows, 3 pieces, or 3 components. So one works in terms of components.

The first row $(x-2)^2$ $x \leq 2$ has an exponent of 2, so it is nonlinear. It also includes a horizontal translation to the right, by two units. This function only exists for $2 \leq x$. This is shown in Graph A, just below.

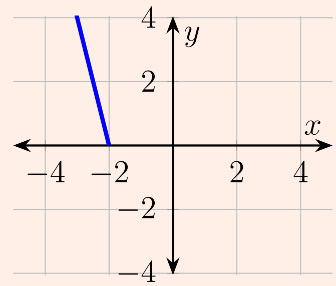
Graph A



Graph B

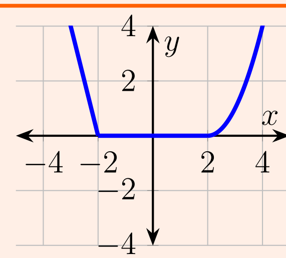


Graph C



In the second row, of the piecewise function, the output is 0 for an input of $-2 \leq x \leq 2$. This function is only valid for $-2 \leq x \leq 2$. This is shown in Graph B, just above.

In the third row, of the piecewise function, the output is $-4x-8$, for the input $x \leq -2$. This linear function has a negative slope. It exists for $x \leq -2$. This is shown in Graph C, just above. The whole piecewise function is graphed here, to the right.



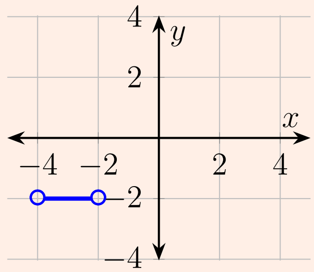
Solved Problem 6.15

Graph the piecewise function $f(x) = \begin{cases} -2 & -4 < x < -2 \\ 0 & -2 < x < 0 \\ 2 & 2 < x < 4 \end{cases}$

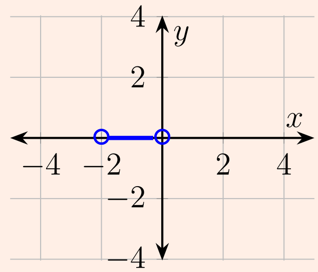
This piecewise function also has 3 rows, 3 pieces, or 3 components. So, again, one works in terms of components.

In the first row, of the piecewise function, the output is -2, for the input $-4 < x < -2$. This component is linear, with a constant output. This function only occurs for the domain $-4 < x < -2$. One can see at $(-4, -2)$ and at $(-2, -2)$ that there are hollow circles, because these endpoints are not included. This occurs for all three pieces of this function. This is shown in Graph A, just below.

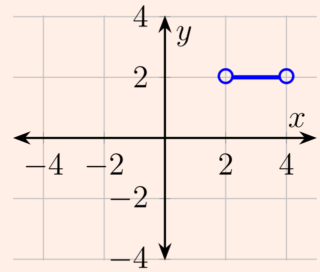
Graph A



Graph B

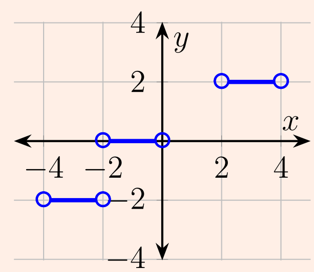


Graph C



In the second row, of the piecewise function, the output is 0, for the input $-2 < x < 0$. This component has a constant output of zero. It exists for the domain $-2 < x < 0$. This is shown in Graph B, just above.

In the third row, of the piecewise function, the output is 2 for input $2 < x < 4$. It is a linear function, with a zero slope. This function exists for $2 < x < 4$. This is shown in Graph C, just above. The whole piecewise function is graphed here, to the right.

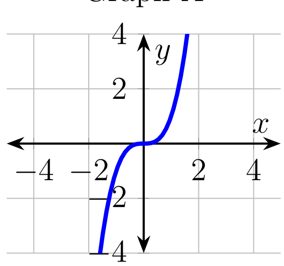


6.7 Functions Problems

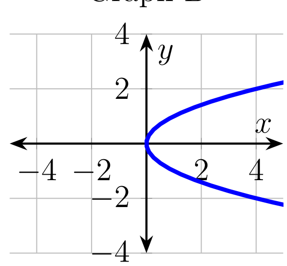
Click on the page number to the right to see the chapter discussion. Click on the solution number to navigate to the solution.

1. Which graphs below display a function? (Page 188) (Solution 1)

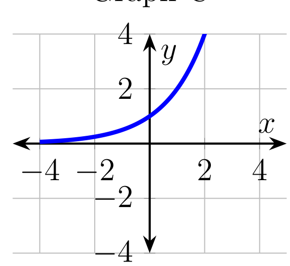
Graph A



Graph B

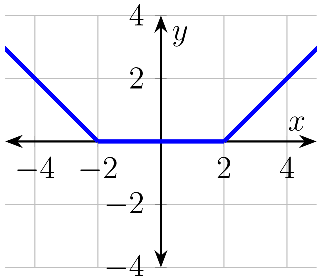


Graph C

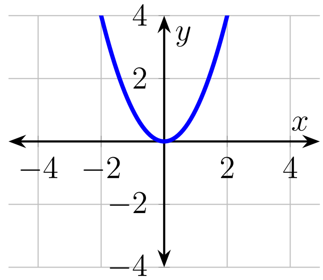


2. Which graph below displays one-to-one mapping? (Page 190) (Solution 2)

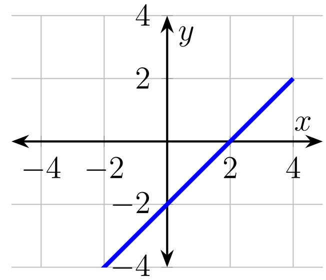
Graph D



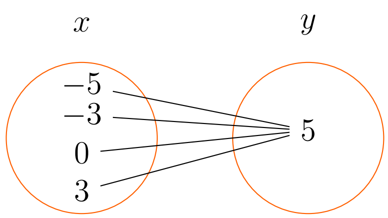
Graph E



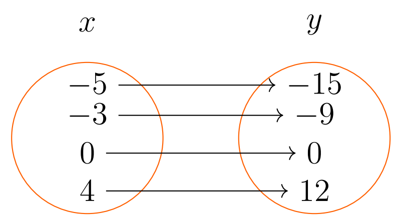
Graph F



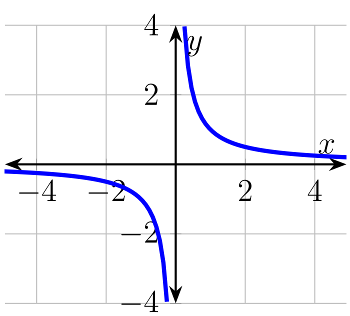
3. What type of mapping is displayed below? (Page 188) (Solution 3)



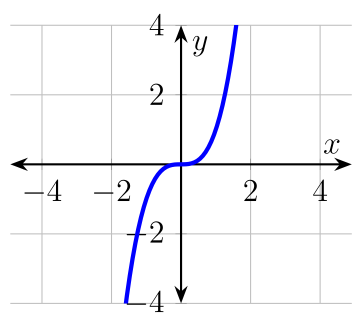
4. What type of mapping is displayed below? (Page 189) (Solution 4)



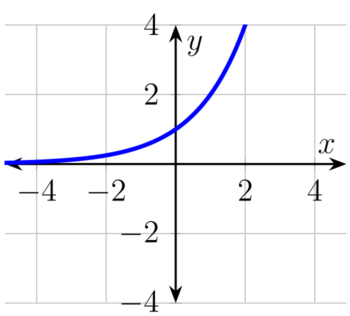
5. Which function corresponds to the graph below? (Page 193) (Solution 5)



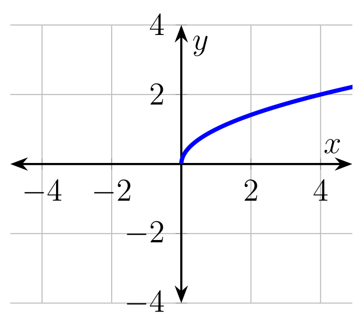
6. Which function corresponds to the graph below? (Page 193) (Solution 6)



7. Which function corresponds to the graph below? (Page 193) (Solution 7)



8. Which function corresponds to the graph below? (Page 192) (Solution 8)



9. Graph the function $f(x) = x^2 + 4$. (Page 195) (Solution 9)

10. Graph the function $f(x) = (x-2)^3$. (Page 195) (Solution 10)

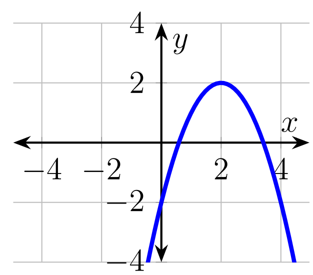
11. When compared to $f(x) = x^2$, which of the following transformations will the function $f(x) = 5x^2$ exhibit? (Page 197) (Solution 11)
    (a) Horizontal stretching
    (b) Horizontal compression
    (c) Vertical Stretching
    (d) Reflection about y-axis

12. When compared to $f(x) = x^2$, which of the following transformations will the function $f(x) = \left(\frac{1}{9}\right) x^2$ exhibit? (Page 198) (Solution 12)
    (a) Horizontal stretching
    (b) Vertical compression
    (c) Vertical Stretching
    (d) Reflection about y-axis

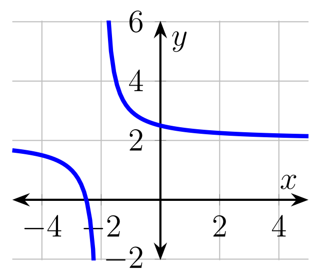
13. When compared to $f(x) = 2^x$, which of the following transformations will the function $f(x) = 2^{4x}$ exhibit? (Page 198) (Solution 13)
    (a) Horizontal compression
    (b) Vertical compression
    (c) Vertical Stretching
    (d) Reflection about y-axis

14. When compared to $f(x) = x^3$, which of the following transformations will the function $f(x) = \left(\frac{1}{3}x\right)^3$ exhibit? (Page 198) (Solution 14)
    (a) Horizontal compression
    (b) Vertical compression
    (c) Horizontal Stretching
    (d) Reflection about y-axis

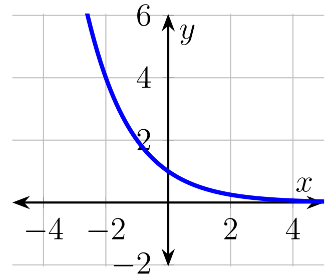
15. Which function corresponds to this graph? (Page 199) (Solution 15)
    (a) $f(x) = x^2 - 2$
    (b) $f(x) = -(x-2)^2 + 2$
    (c) $f(x) = x^2 + 2$
    (d) $f(x) = x^2 - 2$



16. Which function corresponds to this graph? (Page 199) (Solution 16)
    (a) $f(x) = x^2 + 2$
    (b) $f(x) = \dfrac{1}{x-2} + 2$
    (c) $f(x) = 2^{x+2} + 2$
    (d) $f(x) = \dfrac{1}{x+2} + 2$



17. Which function corresponds to this graph? (Page 199) (Solution 17)



(a) $f(x) = 2^{-x}$

(b) $f(x) = x^2$

(c) $f(x) = x^{-2}$

(d) $f(x) = -2^x$

18. Which function transformation corresponds to negating a whole function?
(Page 196) (Solution 18)

19. Solve $(f+g)(x)$ for $f(x) = \left(2x^2+6x+4\right)$ and $g(x) = \left(3x^2-2x-1\right)$.
(Page 200) (Solution 19)

20. Solve $(f-h)(x)$ for $f(x) = \left(9x^2+3x-19\right)$ and $h(x) = \left(6x^2-14\right)$.
(Page 200) (Solution 20)

21. Solve $(f \cdot g)(x)$ for $f(x) = (x+2)$ and $g(x) = (x+4)$.
(Page 201) (Solution 21)

22. Solve $(f \cdot h)(x)$ for $f(x) = (2x+2)$ and $h(x) = (5x-2)$.
(Page 201) (Solution 22)

23. Solve $\left(\frac{f}{h}\right)(x)$ for $f(x) = \left(x^2-3x-4\right)$ and $h(x) = (x+1)$.
(Page 202) (Solution 23)

24. Solve $\left(\frac{f}{h}\right)(x)$ for $f(x) = \left(x^2-2x-4\right)$ and $h(x) = (x-7)$.
(Page 202) (Solution 24)

25. Solve $(f \circ g)(x)$ for $f(x) = \left(9-x^2\right)$ and $g(x) = (x+3)$, and solve for $(f \circ g)(2)$
(Page 203) (Solution 25)

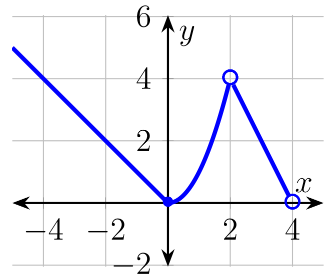
26. Solve $(f \circ h)(x)$ for $f(x) = (5x+3)$ and $h(x) = \left(x^2-1\right)$, and solve for $(f \circ h)(3)$
(Page 203) (Solution 26)

27. For $f(x) = 7x-2$ find $f^{-1}(x)$.
(Page 205) (Solution 27)

28. For $f(x) = \dfrac{3x+4}{11}$ find $f^{-1}(x)$.
(Page 205) (Solution 28)

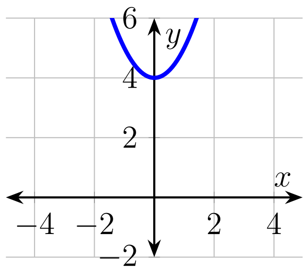
29. Graph the function $f(x) = \begin{cases} x+2 & -4 < x < -2 \\ 2 & -2 < x < 0 \\ -x+4 & 2 < x < 4 \end{cases}$ (Page 207) (Solution 29)

30. Which of the following choices form the piecewise function that is graphed here (select more than one)? (Page 207) (Solution 30)
    (a) $x \quad 0 < x$
    (b) $-2x+8 \quad 2 < x < 4$
    (c) $x^2 \quad 2 \le x < 2$
    (d) $x \quad x \le 0$

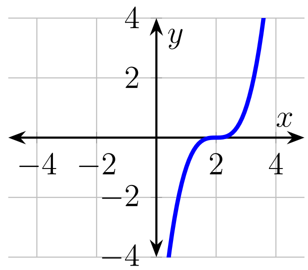


6.8 Functions Solutions

1. Only Graph A and Graph C are functions.
2. Only Graph F displays one-to-one mapping.
3. $f(x) = \frac{1}{x}$
4. $f(x) = x^3$
5. $f(x) = 2^x$
6. $f(x) = \sqrt{x}$
7. $f(x) = x^2 + 4$



8. $f(x) = (x-2)^3$



9. Many-to-one mapping
10. One-to-one mapping
11. (c) Vertical stretching
12. (b) Vertical compression
13. (a) Horizontal compression
14. (c) Horizontal stretching
15. (b) $f(x) = -(x-2)^2 + 2$
16. (d) $f(x) = \frac{1}{x+2} + 2$

17. (a) $f(x) = 2^{-x}$

18. Reflection about the x-axis

19. $5x^2 + 4x + 3$

20. $3x^2 + 3x - 5$

21. $x^2 + 6x + 8$

22. $10x^2 + 6x - 4$

23. The quotient and solution is $x - 4$

24. The quotient is $x + 5$, the remainder is 31. So the full solution is $x + 5 + \dfrac{31}{x-7}$

25. $(f \circ g)(x) = -x^2 - 6x$ then $(f \circ g)(2) = -16$

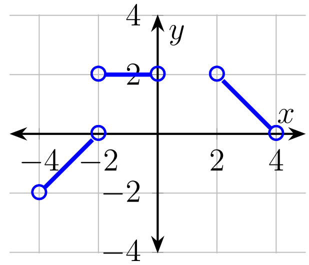
26. $(f \circ g)(x) = 5x^2 - 2$ then $(f \circ g)(3) = 43$

27. $f^{-1}(x) = \dfrac{x+2}{7}$

28. $f^{-1}(x) = \dfrac{11x-4}{3}$

29. The piecewise function and the graph solution...

$$f(x) = \begin{cases} x+2 & -4 < x < -2 \\ 2 & -2 < x < 0 \\ -x+4 & 2 < x < 4 \end{cases}$$



30. The piecewise function consists of 3 components...
(b) $-2x + 8 \quad 2 < x < 4$, (c) $x^2 \quad 0 \leq x < 2$, (d) $-x \quad x \leq 0$

Chapter 7: Exponential & Logarithmic Functions

OVERVIEW

The sections of this chapter are:

7.1 Exponential Growth & Decay
7.2 Visual Exponential Functions
7.3 Exponential Function Transformation
7.4 Logarithms
7.5 Visual Logarithmic Functions
7.6 Logarithmic Function Transformations

Understanding exponents and logarithms is essential for discussing topics like population growth, bacterial experiments, sound levels, finance, or earthquakes. They involve multiplying a number by itself or its inverse and are crucial for forming opinions and engaging in debates on controversial subjects.

OBJECTIVES

By the end of the chapter, a student will be able to:

- Understand discussions regarding exponential growth
- Understand discussions regarding exponential decay
- Apply transformations to exponential functions
- Work with logarithms
- Understand how logarithms and exponents are related
- Work with common and natural logarithms
- Apply transformations to logarithmic functions

7.1 Exponential Growth & Decay

Growth describes the behavior where a value is increasing. The value that is increasing could be the height of a child, the speed of a vehicle, the balance in a bank account, or the human population in a city. Is exponential growth of debt a good thing? The terms linear, nonlinear, and exponential make it possible to understand and describe growth with further clarity.

Fig. 7.1: Linear Growth

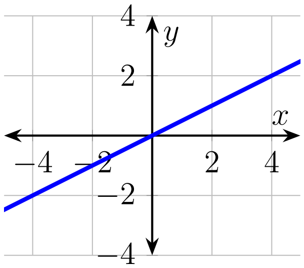


Fig. 7.2: Decay

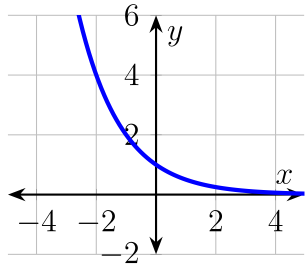
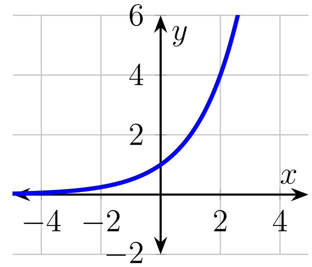


Fig. 7.3: Growth



Figures 7.1 and 7.3 show different types of growth. Figure 7.1 illustrates linear growth, a straight line with a constant slope. In contrast, Figure 7.3 demonstrates exponential growth, where the rate increases rapidly. Figure 7.2 depicts exponential decay, a nonlinear decrease. Exponential growth is significant in various fields, describing rapid expansion in business and variable exponent growth in algebra. Examples of exponential growth are shown below.

$f(x) = 2^x$ $\quad f(x) = 4^x$ $\quad f(x) = 5^x$

For the exponential function $f(x) = 2^x$, the number 2 is the base, and x is the exponent. One can see here that the exponent is variable as opposed to a constant. This can seem complicated at first, but really this equation only states that the number 2 will be multiplied by itself a certain number of times. The following examples reinforce this.

$f(x) = 2^0 = 1$ $\quad f(x) = 2^1 = 2$ $\quad f(x) = 2^2 = 2(2) = 4$ $\quad f(x) = 2^3 = 2(2)(2) = 8$

Since a number is being multiplied by itself, this leads to a very high rate of growth. The examples below demonstrate this for bases other than 2.

$f(x) = 26^0 = 1$ $\quad f(x) = 26^1 = 26$ $\quad f(x) = 26^2 = 26(26) = 676$

$f(x) = 52^0 = 1$ $\quad f(x) = 52^1 = 52$ $\quad f(x) = 52^2 = 52(52) = 2,704$

Exponential growth is based on the basic exponential functions such as $f(x) = 2^x$, but there is another way to describe exponential growth that makes use of a growth rate or multiplier. This approach to exponential growth builds upon the examples shown above.

Definition 7.1 — Exponential Growth. Exponential growth describes a phenomenon or behavior where a value is increasing, and the rate of change is also increasing. This leads to very rapid growth of the output or dependent variable.

$$\boxed{f(x) = a(1+r)^x}$$

a is an initial value $\quad r$ is the growth rate
x is the independent variable

The sum $(1+r)$ is also called the multiplier.

The exponential function $f(x) = 2^x$ has a base of 2. In several examples above whole number bases were used. It may be necessary to use a base that is greater than 1 but also less than

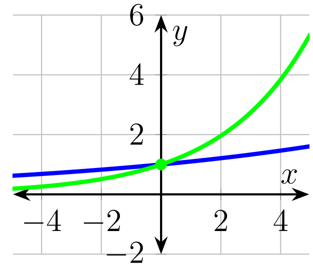
2. The multiplier in the definition, just above, makes it possible to describe a base that is greater than 1, and also less than 2. The value a in the definition above is often called an initial value, and it also applies vertical scaling.

The blue curve shows a growth rate of .1 and the function $f(x) = (1+.1)^x = 1.1^x$

The green curve shows a growth rate of .4 and the function $f(x) = (1+.4)^x = 1.4^x$

This illustrates the effect of different growth rates.

Fig. 7.4: $f(x) = (1+r)^x$



In the graph just above the exponential growth formula $f(x) = b(1+r)^x$ shows up as $f(x) = 1.1^x$. So $b = 1$, and it does not really have an impact on the function. One can see in the graph above, that as the growth rate increases from .1 to .4, the curve displays a higher slope. In the graph above, one can see that the y-intercept is (0, 1), shown in green.

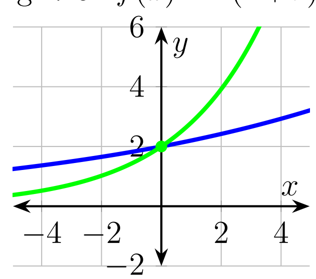
The graph below makes use of the same growth rates .1 and .4, but the whole function is multiplied by 2. As expected, 2 applies vertical scaling or stretching to the exponential function. Notice, below, that the y-intercept is shown in green at (0, 2).

The blue curve shows a growth rate of .1 and the function $f(x) = 2(1+.1)^x$

The green curve shows a growth rate of .4 and the function $f(x) = 2(1+.4)^x$

This illustrates that the initial value 2, also works as a vertical scaling/stretching factor.

Fig. 7.5: $f(x) = 2(1+r)^x$



In the graph above, 2 works as a scaling factor. This value of 2 can also be thought of as an initial value. In the function $f(x) = 2(1+r)^x$, notice that if the growth rate r is zero, then the function becomes $f(x) = 2(1^x)$ which is the same as $f(x) = 2$. So the function becomes a horizontal straight line, because the output would always be zero. With a growth rate of zero, the function never grows, and it stays at the initial value of 2. For the case of the exponential growth function $f(x) = a(1+r)^x$, the value a, can be thought of as an initial value or a vertical scaling factor.

Several examples have demonstrated exponential growth. Exponential decay shows up in the exponential function $f(x) = b^x$ if the base b is between zero and 1. Exponential decay also shows up in the function $f(x) = a(1-r)^x$, where the r is subtracted, and it becomes the decay rate. Recall that, in a function, $f(x)$ can be written as y.

Fig. 7.6: $f(x) = (1/2)^x$

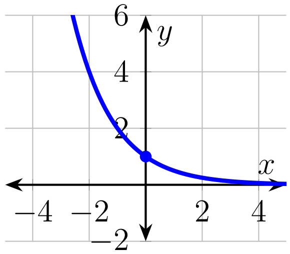


Fig. 7.7: $y = (1-(1/2))^x$

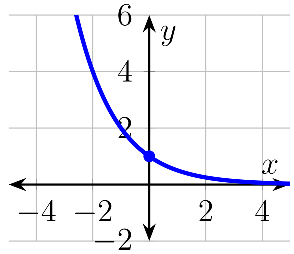
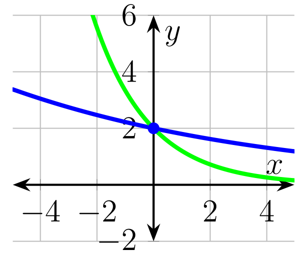


Fig. 7.8: $y = 2(1-r)^x$



In figure 7.6, $f(x) = (1/2)^x$ shows exponential decay with base 1/2. In Figure 7.7, $f(x) = a(1-r)^x$, representing the same function, has an initial value of 1 and a multiplier of 1/2. This equation is equivalent to $f(x) = (1/2)^x$ without an initial value or vertical scaling factor.

Definition 7.2 — Exponential Decay. Exponential decay describes a phenomenon or behavior where a value is decreasing, and the rate of change is also decreasing. This leads to a very rapid decrease or decay of the output or dependent variable.

$$\boxed{f(x) = a(1-r)^x}$$

a is an initial value r is the decay rate
x is the independent variable

The sum $(1-r)$ is also called a multiplier.

Tip If an exponential function $f(x) = b^x$ has a base b that is greater than 1, then it will exhibit exponential growth.

Tip If an exponential function $f(x) = b^x$ has a base b that is between zero and 1, then it will exhibit exponential decay.

Solved Problem 7.1 Will the following functions exhibit growth or decay?
(a) $f(x) = .75^x$ (b) $f(x) = 1.4^x$ (c) $f(x) = .8^x$ (d) $f(x) = 2.4^x$

(a) $f(x) = .75^x$ In the equation $f(x) = a(1-r)^x$ if $a = 1$ and $r = .25$, then it would look like $f(x) = .75^x$. The base of .75, is less than 1, so this function would exhibit $\boxed{\text{decay}}$.

(b) $f(x) = 1.4^x$ In the equation $f(x) = a(1+r)^x$ if $a = 1$ and $r = .4$, then it would look like $f(x) = 1.4^x$. The base of 1.4, is greater than 1, so this function would exhibit $\boxed{\text{growth}}$.

(c) $f(x) = .8^x$ The base of .8 is less than 1, so this function would exhibit $\boxed{\text{decay}}$.

(d) $f(x) = 2.4^x$ The base of 2.4 is greater than 1, so this function would exhibit $\boxed{\text{growth}}$.

■

Solved Problem 7.2 For the function $f(x) = 5(1 - .3)^x$:

(a) Identify if it describes growth or decay.

(b) Identify the growth or decay rate.

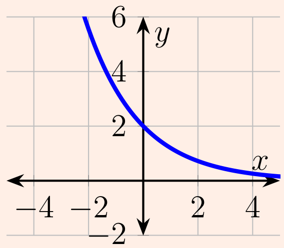
(c) Identify the initial value.

The rate is $\boxed{-3}$, and since it is subtracted, this function describes decay. It will have a decay rate, not a growth rate. In this function, the decay rate is .3, and the initial value is 5. $\boxed{\text{Final Answer (a) decay} \quad \text{(b) } -.3 \quad \text{(c) } 5}$ ■

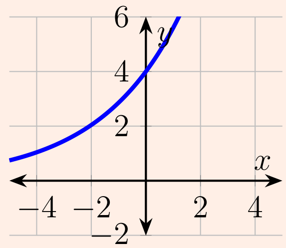
Solved Problem 7.3 Match the graphs below with the corresponding functions listed here

(a)$f(x) = 2(1 - .1)^x$ (b)$f(x) = 1(1 - .1)^x$ (c)$f(x) = 4(1 + .4)^x$ (d)$f(x) = 2(1 - .4)^x$

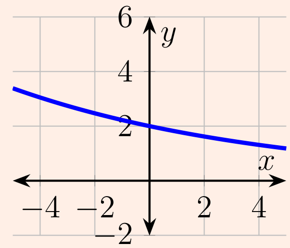
Graph A



Graph B



Graph C



Only Graph B shows exponential growth. Graphs A and C, show exponential decay. Only one of the functions listed shows a positive growth rate. That would be function (c)$f(x) = 4(1 + \boxed{.4})^x$ with its growth rate of .4, and the other three listed functions have negative growth rates, which are decay rates. So, Graph B corresponds to function (c).

This leaves two graphs to consider, and this leaves 3 listed functions to consider, meaning function (a), (b), and (d). Both Graph A and Graph C have a y-intercept of 2. This y-intercept should correspond to the initial value of the function. Only the listed functions (a) and (d) have an initial value of 2. Both functions (a) and (d) have a rate of decay. It will be necessary to consider the rates of decay.

(a)$f(x) = 2(1 \boxed{-.1})^x$ This rate of decay of .1 would correspond to the graph with a lesser inclination or slope. Of Graph A and Graph C, this would be Graph C. Graph C corresponds to function (a).

(d)$f(x) = 2(1 \boxed{-.4})^x$ This rate of decay of -4 would then correspond to the other graph with a y-intercept of 2. Graph A corresponds to function (d).

Graph A - (d) Graph B - (c) Graph C - (a) ■

Tip The exponential function $f(x) = a(1+r)^x$ has a rate that is added, so it will exhibit exponential growth.

Tip The exponential function $f(x) = a(1-r)^x$ has a rate that is subtracted, so it will exhibit exponential decay.

All this is simplified greatly by understanding that there is simply a rate. A positive rate simply describes growth. A negative rate describes decay.

7.2 Visual Exponential Functions

In order to reinforce recognition and understanding of exponential functions it is necessary to describe these functions in more specific ways.

- Growth/decay rate
- Domain
- Range
- Asymptotes
- Y-intercept
- One-to-one mapping

Exponential growth functions have increasing positive slopes, while exponential decay functions have decreasing negative slopes. Both $f(x) = 2^x$ and $f(x) = a(1+r)^x$ are exponential functions. One needs to be able to see how these are related.

Solved Problem 7.4 Identify the growth or decay rate in these functions

(a) $f(x) = 2.3^x$ (b) $f(x) = 3.4^x$ (c) $f(x) = .8^x$

These functions may appear different from the format $f(x) = a(1+r)^x$ or $f(x) = a(1-r)^x$, but they have the same information.

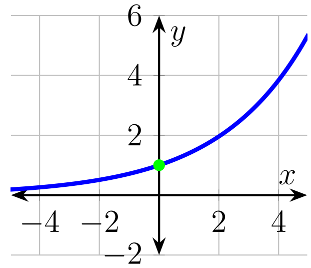
(a) $2.2^x = 2(1+\boxed{.1})^x$ .1 is added, so $\boxed{\text{the growth rate is .1}}$

(b) $3.9^x = 3(1+.3)^x$ .3 is added, so $\boxed{\text{the growth rate is .3}}$

(c) $.8^x = 1(1-.2)^x$ .2 is subtracted, so $\boxed{\text{the decay rate is -.2}}$ ■

The domain, range, y-intercept, and possible asymptotes make it possible to clearly describe growth and decay functions.

Fig. 7.9: $f(x) = 2(1+.4)^x$



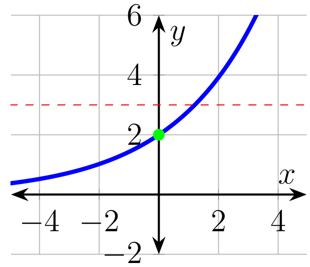
The domain is $(-\infty, \infty)$ or $\{x \mid x \in \mathbb{R}\}$

The range is $(0, \infty)$ or $\{x \mid 0 < x\}$

The y-intercept is found at (0, 1), shown in green.

One asymptote exists at $y = 0$

Fig. 7.10: $f(x) = 2(1+.4)^x$



The domain is $(-\infty, \infty)$ or $\{x \mid x \in \mathbb{R}\}$

The range is $(0, \infty)$ or $\{x \mid 0 < x\}$

The y-intercept is found at (0, 2), shown in green

One asymptote exists at $y = 0$

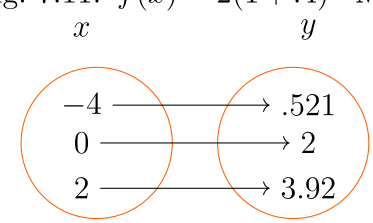
For the graph above, a horizontal line never touches more than one point at a time, so this function has one-to-one mapping. The mapping diagram below shows that for the function $f(x) = 2(1+.4)^x$, each specific input there is a specific output.

If $x = -4$, then $y = .521$

If $x = 0$, then $y = 2$

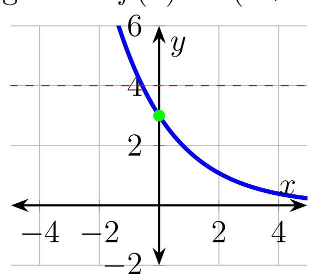
If $x = 2$, then $y = 3.92$

Fig. 7.11: $f(x) = 2(1+.4)^x$ Map



For the exponential decay function, shown below, the slope is clearly negative, and the y-intercept of 3, instead of 1, as shown below.

Fig. 7.12: $f(x) = 2(1+.4)^x$



The domain is $(-\infty, \infty)$ or $\{x \mid x \in \mathbb{R}\}$

The range is $(0, \infty)$ or $\{x \mid 0 < x\}$

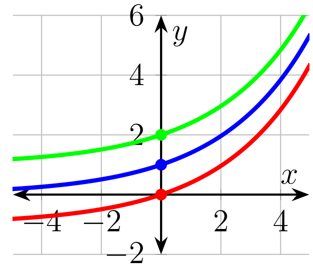
The y-intercept is found at (0, 3), shown in green

One asymptote exists at $y = 0$

7.3 Exponential Function Transformation

The graph below illustrates vertical translation. Vertical translation refers to either shifting up or down. For the graph below, a constant c, is added to or subtracted from the function. Translation or a shift can occur in a vertical manner, and translation can occur in a horizontal manner.

Fig. 7.13: $f(x) = 1(1+.4)^x + c$

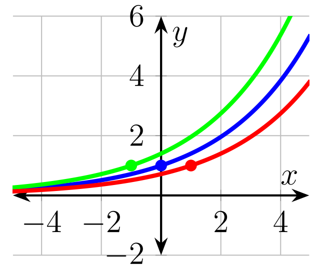


The blue curve shows $f(x) = 1(1+.4)^x$

The green curve shows $f(x) = (1+.4)^x + 1$. The constant 1, is added. The curve shifts up.

The red curve shows $f(x) = (1+.4)^x - 1$. The constant 1, is subtracted. The curve shifts down.

Fig. 7.14: $f(x) = 1(1+.4)^{x+c}$



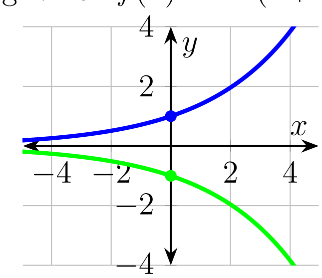
The blue curve shows $f(x) = 1(1+.4)^x$

The green curve shows $f(x) = (1+.4)^{x+1}$. The constant 1, is added to the variable. The curve shifts to the left.

The red curve shows $f(x) = (1+.4)^{x-1}$. The constant 1, is subtracted from the variable. The curve shifts to the right.

Reflection can occur about the x-axis, and reflection can also occur about the y-axis. The graph below illustrates reflection, about the x-axis. A constant -1, is multiplied by the whole function. The y-intercept responds to this horizontal reflection. The y-intercept becomes (0, -1).

Fig. 7.15: $f(x) = -1(1+.4)^x$



The blue curve shows $f(x) = -1(1+.4)^x$

The green curve shows $f(x) = -(1+.4)^x$. The whole function is negated. The curve reflects about the x-axis.

The graph below illustrates reflection about the y-axis, applied to the exponential function $f(x) = 1(1+.4)^x$. A constant -1, is multiplied by the variable x, within. The y-intercept remains the same after the reflection.

The blue curve shows $f(x) = 1(1+.4)^x$

The red curve shows $f(x) = (1+.4)^{(-1x)}$. Only the variable within, is negated. The curve shifts about the y-axis.

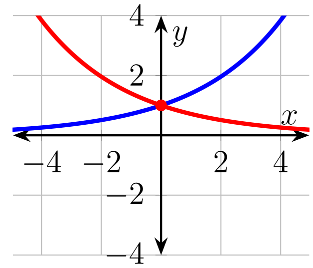


Fig. 7.16: $f(x) = -1(1+.4)^x$

The graph below illustrates vertical scaling. Vertical scaling refers to either vertical stretching or vertical compression.

The blue curve shows $f(x) = 1(1+.4)^x$

The green curve shows $f(x) = 2(1+.4)^x$. The constant c, is the whole number 2, vertical stretching occurs.

The red curve shows $f(x) = (1/2)(1+.4)^x$. The constant c, is the fraction (1/2), vertical compression occurs.

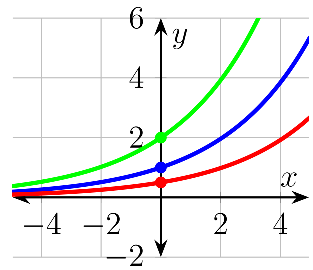


Fig. 7.17: $f(x) = c(1+.4)^x$

The graph, below, illustrates horizontal scaling. Horizontal scaling refers to either horizontal stretching or horizontal compression. For the graph below, the variable x in the exponent is multiplied by a constant c, that can also be called a horizontal scaling factor.

The blue curve shows $f(x) = 2(1+.4)^x$

The green curve shows $f(x) = 2(1+.4)^{2x}$. The constant c, is the whole number 2, horizontal compression occurs.

The red curve shows $f(x) = 2(1+.4)^{(1/2)x}$. The constant c, is the fraction (1/2), horizontal stretching occurs.

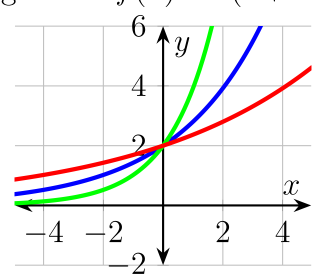


Fig. 7.18: $f(x) = 2(1+.4)^{cx}$

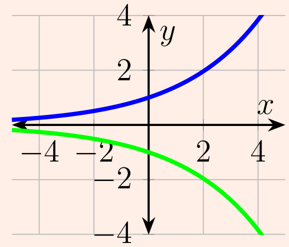
Solved Problem 7.5 Graph the function $f(x) = -(1+.4)^x + 3$.

This exponential function involves a horizontal reflection and a vertical translation.

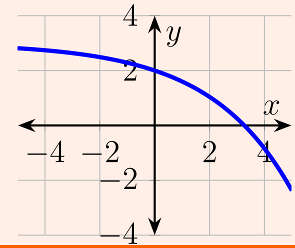
The blue curve in Graph A is $f(x) = (1+.4)^x$. This is the starting function.

The green curve is $f(x) = -1(1+.4)^x$.

Graph A



Graph B



The green curve, above, shows the first transformation, which is a reflection about the x-axis, and the function $f(x) = -1(1+.4)^x$. The blue curve in Graph B, shows the second transformation, and the function $f(x) = -1(1+.4)^x + 3$, which adds a vertical translation/shift up, by 3 units. Graph B shows the final answer. ■

7.4 Logarithms

Both exponents and logarithms stem from the simple concept of a number that is multiplied by itself. At this point, the following exponential notation has been reinforced in different ways.

$3^2 = 3(3) = 9$ The base is 3, and the exponent is 2.

Exponential notation, as shown, just above, has been practiced in different ways. The concept of the base is also used with logarithms. A logarithm describes the exponent that will be needed to produce a specific result. Logarithms involve new terms, but the concepts are nearly the same.

$\log_3 9 = 2$ The logarithm is 2. It is the logarithm of 9, with base 3.

A logarithmic expression can be thought of as a question that is asking about an exponent. The examples below demonstrate logarithms.

If the base is 2, what exponent produces 8?	$\log_2 8 = 3$
If the base is 2, what exponent produces 16?	$\log_2 16 = 4$
If the base is 3, what exponent produces 27?	$\log_3 27 = 3$
If the base is 10, what exponent produces 100?	$\log_{10} 100 = 2$

For example, the expression $\log_2 8 = 3$ is read as the logarithm, base 2 of 8 is 3. In a shortened version, it is read as the log, base 2 of 8 is 3.

Definition 7.3 — Logarithm. A logarithm is the number of times that a base needs to be multiplied by itself in order to produce a specific number.

$\boxed{\log_b x = n}$ b is the base x is the argument
n is the logarithm or exponent

The base b is greater than one.

Solved Problem 7.6 For the logarithmic expression $\log_2 32 = 5$, identify the

(a) base (b) argument (c) logarithm

Final Answer (a) 2 (b) 32 (c) 5 ▪

Tip For the logarithm $\log_b x = n$, the argument is x

Solved Problem 7.7 Evaluate the following logarithmic expressions.

(a) $\log_{10} 1000$ (b) $\log_4 64$ (c) $\log_5 25$ (d) $\log_2 64$

(a) $10 \times 10 \times 10 = 1000$ so $\log_{10} 1000 = \boxed{3}$ (b) $4 \times 4 \times 4 = 64$ so $\log_4 64 = \boxed{3}$

(c) $5 \times 5 = 25$ so $\log_5 25 = \boxed{2}$ (d) $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$ so $\log_2 64 = \boxed{6}$ ▪

Logarithmic Laws

1. Log of 1 $\log_b 1 = 0$
2. Log base b of b $\log_b b = 1$
3. Product $\log_b xy = \log_b x + \log_b y$
4. Ratio $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$
5. Change of bases $\log_b x = \log_c x \log_b c$
6. Power $\log_b x^y = y \log_b x$
7. Root $\log_b \sqrt[y]{x} = \frac{1}{y} \log_b x$

The product law for logarithms is demonstrated below.

$$\log_2 8 = \boxed{3} \qquad \log_2 2(4) = \log_2 2 + \log_2 4 = 1 + 2 = \boxed{3}$$

The ratio law for logarithms is demonstrated below.

$$\log_{10} 100 = \boxed{2} \qquad \log_{10}\left(\frac{1000}{10}\right) = \log_{10} 1000 - \log_{10} 10 = 3 - 1 = \boxed{2}$$

The power law makes it possible to remove an exponent from the argument of the logarithm. Below, the exponent of 2 in the argument, is moved outside the logarithm, and it is then multiplied by the logarithm. The power law for logarithms is demonstrated below.

$$\log_2\left(4^2\right)=\log_2 16=\boxed{4} \qquad \log_2\left(4^2\right)=2\left(\log_2 4\right)=2(2)=\boxed{4}$$

Similarly, The root law makes it possible to remove a radical from the argument of the logarithm.

$$\log_2 \sqrt[2]{16}=\log_2 4=\boxed{2} \qquad \log_2 \sqrt[2]{16}=\frac{1}{2}\left(\log_2 16\right)=\frac{1}{2}(4)=\boxed{2}$$

A logarithm may have a base of 2, but one may need or prefer a base of 4. The change of bases law makes it possible to change the base from 2 to 4. The change of bases law is demonstrated below.

$$\log_2 16=\boxed{4} \qquad \log_2 16=\log_4 16\left(\log_2 4\right)=2(2)=\boxed{4}$$

Solved Problem 7.8 Apply the logarithm laws to evaluate the following logarithmic expressions. (a) $\log_3 81$ (b) $\log_5 50-\log_5 2$ (c) $\log_{10} 200-\log_{10} 2$

(a) $\log_3 81=\log_3 9(9)=\log_3 9+\log_3 9=2+2=\boxed{4}$

(b) $\log_5 50-\log_5 2=\log_5\left(\frac{50}{2}\right)=\log_5 25=\boxed{2}$ because $5^2=25$

(c) $\log_{10} 200-\log_{10} 2=\log_{10}\left(\frac{200}{2}\right)=\log_{10} 100=\boxed{2}$ because $10^2=100$ ■

In a logarithm, if the base is 10, this is called a common logarithm, and examples of this are shown below.

$$f(x)=\log_{10} 10=1 \qquad f(x)=\log_{10} 100=2 \qquad f(x)=\log_{10} 1000=3$$

For the case of common logarithms, where the base is 10, the base is usually omitted when writing the logarithm.

$$f(x)=\log 10=1 \qquad f(x)=\log 100=2 \qquad f(x)=\log 1000=3$$

Solved Problem 7.9 Solve the following expressions... (a) $\log 10000$ (b) $\log 100000$

(a) $\log 10,000=\log_{10}(10,000)=\boxed{4}$ This is because $10^4=10\times10\times10\times10=10,000$. Notice, above, that the solution of 4 is equal to the number of zeros in 10,000.

(b) $\log 100,000=\log_{10}(100,000)=\boxed{5}$ This is because $10^5=10\times10\times10\times10\times10=100,000$. Notice, above, that the solution of 5 is equal to the number of zeros in 100,000.

■

In an exponential function one may see the base e or 2.718. The exponential function $f(x) = e^x$ is illustrated below.

$f(x) = e^x$ is the natural exponential function

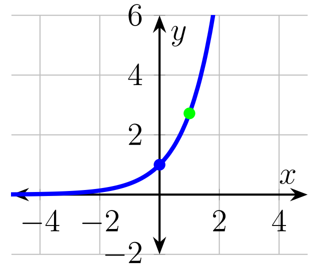
It displays exponential growth.

The slope at the blue point $x = 0$ is $e^{(0)} = 1$

The slope at the green point $x = 1$ is $e^{(1)} = e$

The slope at any point x on the curve is e^x

Fig. 7.19: $f(x) = e^x$



Euler's number, approximately 2.718, is crucial across various fields due to its ubiquitous presence in mathematics, engineering, physics, finance, and chemistry. Named after Leonhard Euler, its applications range from logarithms to exponential growth, making it indispensable in diverse disciplines

- For $f(x) = e^x$ the slope is simply and always e^x
- Euler's number can be used to describe a circle

Solved Problem 7.10 Find the slope of the following exponential functions.

(a) The slope of $f(x) = e^x$ at $x = -2$ (b) $f(x) = e^x$ at $x = 2$

(a) The slope of $f(x) = e^x$ at $x = -2$ is $e^{(-2)}$. This can be plugged into a calculator, which leads to the solution .135 or, we can leave it as the exact solution $\boxed{e^{(-2)}}$

(b) The slope of $f(x) = e^x$ at $x = 2$ is $\boxed{e^{(2)}}$. Using the calculator leads to the approximation 7.389, but $e^{(2)}$ is a correct and quicker solution. ▪

A natural logarithm is a logarithm where the base is Euler's number e. Examples of natural logarithms are shown below.

$$\log_e 1 = 0 \quad \log_e e = 1 \qquad \text{can be rewritten as} \qquad \ln 1 = 0 \quad \ln e = 1$$

As shown above, for the case of natural logarithms log is rewritten ln, where ln means natural log.

The natural logarithm is juust like a regular logarithm. The examples below reinforce this.

If the base is e, what exponent produces 1?	$\ln 1 = 0$
If the base is e, what exponent produces e?	$\ln e = 1$

Tip For natural logarithms, the base of e is usually omitted.

The similarities between exponential notation and logarithms have been emphasized. These similarities exist because the inverse of an exponential function is a logarithmic function.

Fig. 7.20: $f(x) = 2^x$

Input $x = 3$ $\longrightarrow$ $\boxed{f(x) = 2^x}$ $\longrightarrow$ **Output** $f(x) = 8$

Fig. 7.21: $f^{-1}(x) = \log_2 x$

Input $x = 8$ $\longrightarrow$ $\boxed{f^{-1}(x) = \log_2 x}$ $\longrightarrow$ **Output** $f(x) = 3$

The images above, clearly show that a logarithm is the inverse of an exponential function. One can clearly see that this causes the input and output to reverse.

$2^x \quad \rightarrow \quad \boxed{\log}\,_2 x \qquad\qquad f(x) = 2^x \quad \rightarrow \quad f^{-1}(x) = \log_2 x$

Examples of finding the inverse of an exponential function are shown below. Recall that, for the function $f(x)$ the inverse is $f^{-1}(x)$.

$f(x) = 3^x \quad 3^x \rightarrow \boxed{\log}\,_3 x \qquad\qquad f^{-1}(x) = \log_3 x$

$f(x) = 5^x \quad 5^x \rightarrow \boxed{\log}\,_5 x \qquad\qquad f^{-1}(x) = \log_5 x$

The same concepts can be applied to common logarithms and natural logarithms, as shown below. For a natural logarithm, the expected log is rewritten as ln.

$f(x) = 10^x \quad 10^x \rightarrow \boxed{\log}\,_{10} x \qquad\qquad f^{-1}(x) = \log_{10} x \;$ or $\; f^{-1}(x) = \log x$

$f(x) = e^x \quad e^x \rightarrow \boxed{\log}\,_e x \qquad\qquad f^{-1}(x) = \log_e x \;$ or $\; f^{-1}(x) = \ln x$

What about finding the inverse of a logarithmic function?

$f(x) = \log_6 x \quad f(x) = \log_6 x \rightarrow \boxed{\cancel{\log}}\,_6 x \qquad\qquad f^{-1}(x) = 6^x$

$f(x) = \log_8 x \quad f(x) = \log_8 x \rightarrow \boxed{\cancel{\log}}\,_8 x \qquad\qquad f^{-1}(x) = 8^x$

It is often necessary to go back and forth with exponentials and logarithms in this manner.

Solved Problem 7.11 Find the inverse of each function that follows.

(a) $f(x) = \log_7 x$ (b) $f(x) = \log_4 x$ (c) $f(x) = 13^x$ (d) $f(x) = 19^x$

(a) $f(x) = \cancel{\log}\ _7 x \rightarrow \boxed{f^{-1}(x) = 7^x}$

(b) $f(x) = \cancel{\log}\ _4 x \rightarrow \boxed{f^{-1}(x) = 4^x}$

(c) $f(x) = 13^x \rightarrow \boxed{f^{-1}(x) = \log_{13} x}$

(d) $f(x) = 19^x \rightarrow \boxed{f^{-1}(x) = \log_{19} x}$

Given a logarithmic expression/equation, one may need to isolate x. This makes it necessary to remove the log component from the logarithmic expression, in order to isolate the argument x.

$$\log_b x \text{ The base is } b \rightarrow b^{\log_b x} = x \rightarrow \cancel{b}^{\cancel{\log_b}\ x} = x$$

As we seen above, to remove the logarithmic notation we need to apply the exponential of the same base as the give logarithm.

$$\log_2 x \text{ The base is } 2 \rightarrow 2^{\log_2 x} \rightarrow \cancel{2}^{\cancel{\log_2}\ x} = x$$

Again, the rightmost equation, just above, $\cancel{2}^{\cancel{\log_2}\ x} = x$, has the logarithmic expression, $\log_2 x$ in the exponent. This exponent means that there is some exponent that is applied to 2, in order to produce x. As was just stated, if such an exponent is applied to 2, then the result is x.

$$b^x \text{ The base is } b \rightarrow \log_b b^x \rightarrow \cancel{\log_b}\ \cancel{b}^x = x$$

$$2^x \text{ The base is } 2 \rightarrow \log_2 2^x \rightarrow \cancel{\log_2}\ \cancel{2}^x = x$$

Recall that, if the base is "missing" in a logarithm, then the base is 10. For the natural logarithm ln, the base is Euler's number e.

Tip The natural log is written $\ln x$, and it means $\log_e x$

Solved Problem 7.12 Solve the following logarithmic equations for x.

(a) $\log_2 x = 3$ (b) $\log_4 x = 2$ (c) $\ln x = 1$ (d) $\log_{10} x = 3$

(a) If $\log_2 x = 3$ the base is 2 $\rightarrow 2^{\log_2 x} = 2^3 \rightarrow \cancel{2}^{\cancel{\log_2}\ x} = 2^3$ $\boxed{x = 8}$

(b) If $\log_4 x = 2$ the base is 4 $\rightarrow 4^{\log_4 x} = 4^2 \rightarrow \cancel{4}^{\cancel{\log_4}\ x} = 4^2$ $\boxed{x = 16}$

(c) If $\ln x = 1$ $\boxed{\text{the base is } e}$ $\rightarrow e^{\log_e x} = e^1 \rightarrow \cancel{e}^{\cancel{\log_e}\ x} = e$ $\boxed{x = e}$

(d) If $\log x = 3$ $\boxed{\text{the base is 10}}$ $\rightarrow$ $10^{\log_{10} x} = 10^3$ $\rightarrow$ $\cancel{10}^{\cancel{\log_{10}} x} = 1000$ $\boxed{x = 1000}$ ■

On the contrary, if the exponential equation is given, one will apply the logarithm of the same base as the exponential, on both sides of the equation, in order to remove the exponential notation.

Solved Problem 7.13 Solve the following exponential equations for x.

(a) $3^x = 27$ (b) $5^x = 25$

(a) If $3^x = 27$ the base is 3 $\rightarrow$ $\log_3 3^x = \log_3 27$ $\rightarrow$ $\cancel{\log_3}\, \cancel{3}^x = 3$ $\boxed{x = 3}$

(b) If $5^x = 25$ the base is 5 $\rightarrow$ $\log_5 5^x = \log_5 25$ $\rightarrow$ $\cancel{\log_5}\, \cancel{5}^x = 2$ $\boxed{x = 2}$ ■

7.5 Visual Logarithmic Functions

One should be able to recognize the general nature of a logarithmic function. In addition, it should be possible to describe a logarithmic function, with regard to the following aspects.

- Domain
- Range
- Asymptotes
- One-to-one mapping

It is helpful to see how a logarithmic function relates to an exponential function.

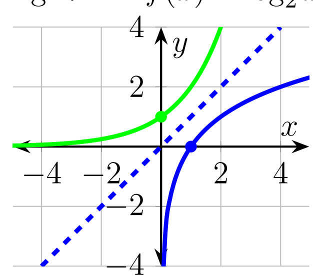
The logarithmic function $f(x) = \log_2 x$ is in blue.

The exponential function $f(x) = 2^x$ is in green.

The blue, dashed line at $y = x$ is an axis of symmetry.

Both options exhibit growth.

Fig. 7.22: $f(x) = \log_2 x$



Knowing that the exponential and the logarithmic functions are inverses, the exponential function's (0, 1) becomes the logarithmic function's (1, 0), showing the reversal of input and output. Exponential growth is convex, while logarithmic growth is concave. The logarithmic function approaches but doesn't touch the y-axis at x=0.One can also see that the domain of the blue, logarithmic function is $(0, -\infty)$, $0 < x$, or $\{x \mid 0 < x\}$. The range is all real numbers, $(-\infty, \infty)$, or $\{x \mid x \in \mathbb{R}\}$.

Fig. 7.23: $f(x) = \log_2 x$

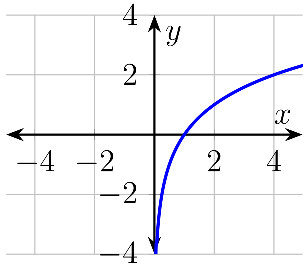


Fig. 7.24: $f(x) = \ln x$

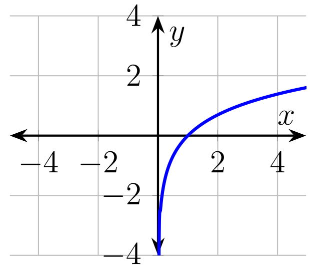
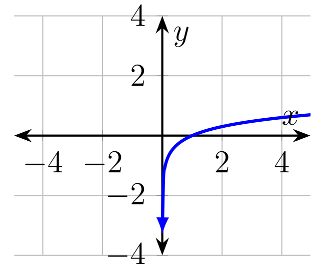


Fig. 7.25: $f(x) = \log x$



The three functions shown above are $f(x) = \log_2 x$, $f(x) = \log_e x$, and $f(x) = \log_{10} x$. This is because $\ln x = \log_e x$, and $\log x = \log_{10} x$.

One can see, again, below, that as the base of a logarithm increases, the curve becomes more of a sharp corner.

Fig. 7.26: $f(x) = \log_2 x$

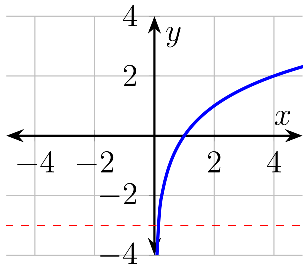


Fig. 7.27: $f(x) = \log_3 x$

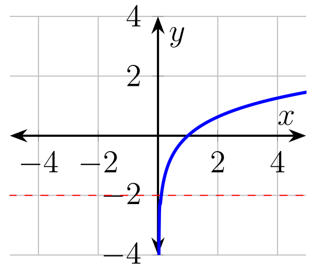
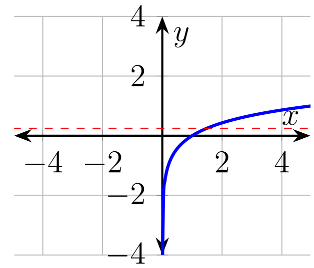


Fig. 7.28: $f(x) = \log_5 x$



7.6 Logarithmic Function Transformations

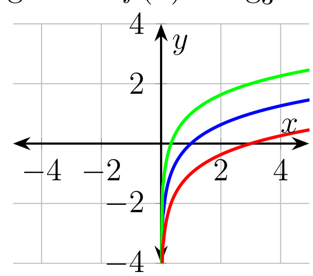
The graph below demonstrates vertical translations or shifts, applied to logarithmic functions.

$f(x) = \log_3 x$ is shown in blue.

A shift up $f(x) = \log_3 x + 1$ is shown in green.

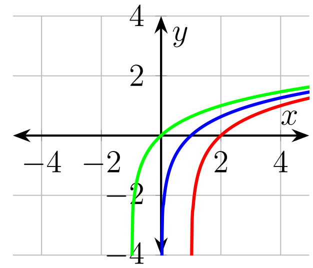
A shift down $f(x) = \log_3 x - 1$ is shown in red.

Fig. 7.29: $f(x) = \log_3 x + c$



The graph below demonstrates horizontal translations or shifts, applied to logarithmic functions.

Fig. 7.30: $f(x) = \log_3(x + c)$



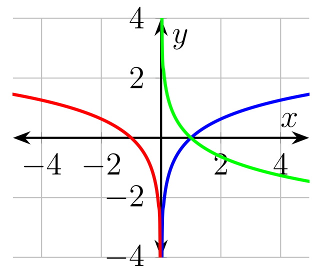
$f(x) = \log_3 x$ is shown in blue.

A shift left $f(x) = \log_3(x + 1)$ is shown in green.

A shift right $f(x) = \log_3(x - 1)$ is shown in red.

The graph below demonstrates reflections, applied to logarithmic functions.

Fig. 7.31: $f(x) = \pm\log_3 \pm x$



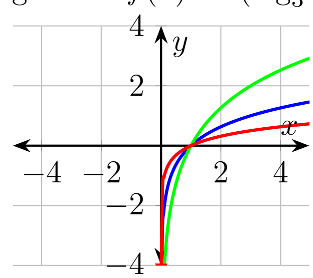
$f(x) = \log_3 x$ is shown in blue.

Reflection about the x-axis $f(x) = -\log_3(x)$ is shown in green.

Reflection about the y-axis $f(x) = \log_3(-x)$ is shown in red.

Scaling can occur in a vertical manner, and scaling can occur in a horizontal manner. The graph below demonstrates vertical stretching/compression, applied to logarithmic functions.

Fig. 7.32: $f(x) = c(\log_3 x)$



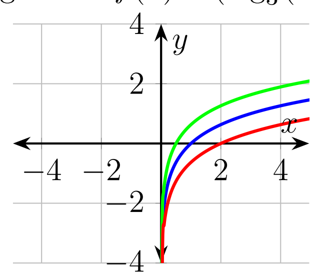
$f(x) = \log_3 x$ is shown in blue.

Vertical stretching $f(x) = 2\log_3(x)$ is shown in green.

Vertical compression $f(x) = \dfrac{1}{2}\log_3(x)$ is shown in red.

The graph below demonstrates horizontal stretching/compression, applied to logarithmic functions.

Fig. 7.33: $f(x) = (\log_3(cx))$



$f(x) = \log_3 x$ is shown in blue.

Horizontal compression $f(x) = \log_3(2x)$ is shown in green.

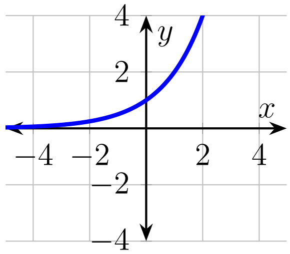
Horizontal stretching $f(x) = \log_3((1/2)(x))$ is shown in red.

7.7 Exponential & Logarithmic Functions Problems

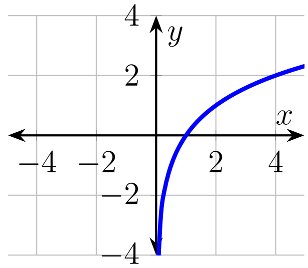
Click on the page number to the right to see the chapter discussion. Click on the solution number to navigate to the solution.

1. Which graph below displays exponential growth? (Page 215) (Solution 1)

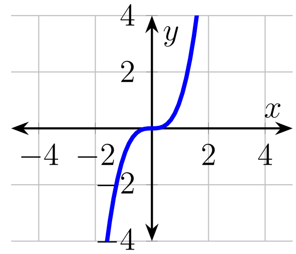
Graph A



Graph B

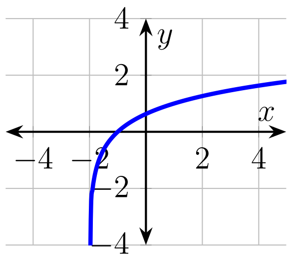


Graph C

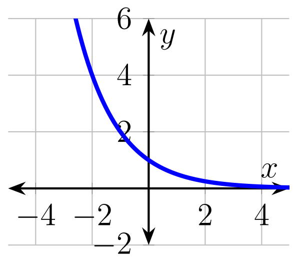


2. Which graph below displays exponential decay? (Page 217) (Solution 2)

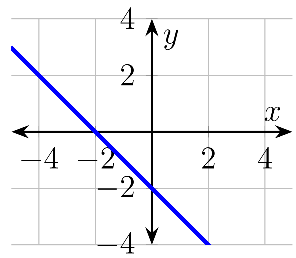
Graph D



Graph E

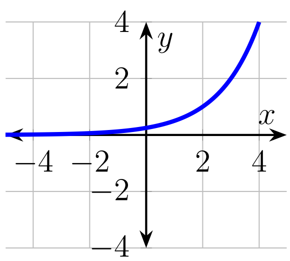


Graph F

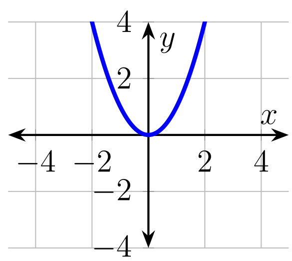


3. Which graph below displays a logarithmic function? (Page 230) (Solution 3)

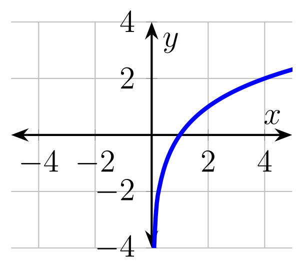
Graph G



Graph H



Graph I



4. In the function $f(x) = 3(1+.2)^x$ identify the initial value and the growth rate. (Page 219) (Solution 4)

5. In the function $f(x) = (1/2)(1-.3)^x$ identify the initial value and the decay rate. (Page 219) (Solution 5)

6. In the function $f(x) = .75(1-.6)^x$ identify the initial value and the decay rate. (Page 219) (Solution 6)

7. Will the function $f(x) = .7^x$ display growth or decay?
(Page 218) (Solution 7)

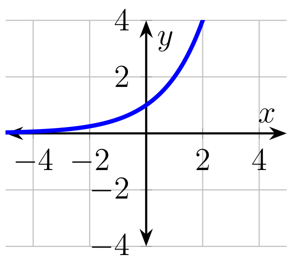
8. Will the function $f(x) = 1.2^x$ display growth or decay?
(Page 218) (Solution 8)

9. Will the function $f(x) = 2.5(1-.2)^x$ display growth or decay?
(Page 219) (Solution 9)

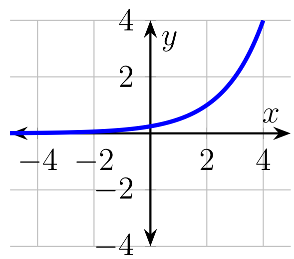
10. Will the function $f(x) = 5(1+.1)^x$ display growth or decay?
(Page 219) (Solution 10)

11. Match the functions listed here with the corresponding graphs below.
$f(x) = 2^x - 2$ $\quad f(x) = 2^{x-2}$ $\quad f(x) = 2^x$. (Page 222) (Solution 11)

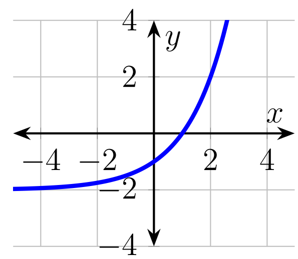
Graph A



Graph B

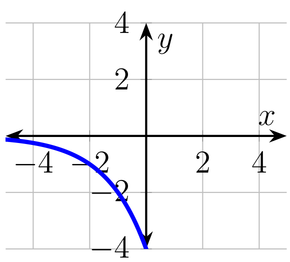


Graph C

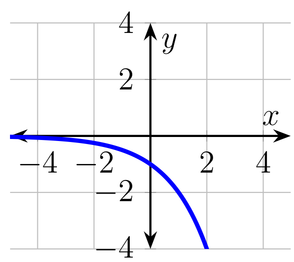


12. Match the functions listed here with the corresponding graphs below.
$f(x) = 2^x$ $\quad f(x) = -2^{x+2}$ $\quad f(x) = -2^x$ (Page 223) (Solution 12)

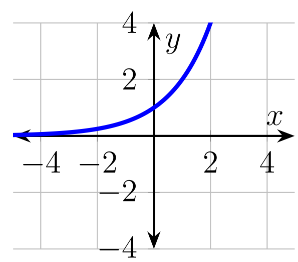
Graph D



Graph E

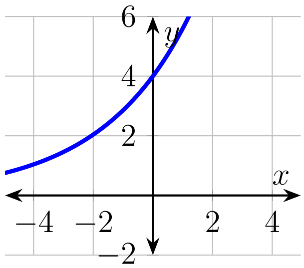


Graph F

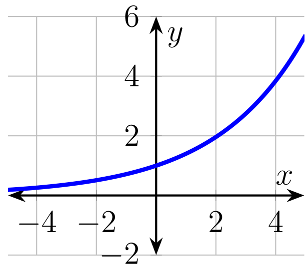


13. Match each function with its corresponding graph. (Page 223) (Solution 13)
$f(x) = (1+.4)^x$ $\quad f(x) = 4(1+.4)^x$ $\quad f(x) = (1/2)(1+.4)^x$

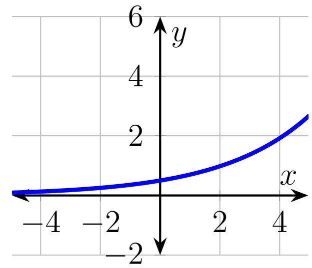
Graph G



Graph H



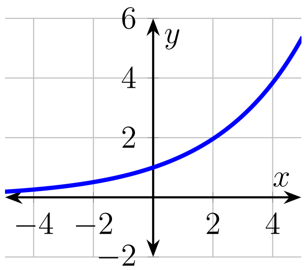
Graph I



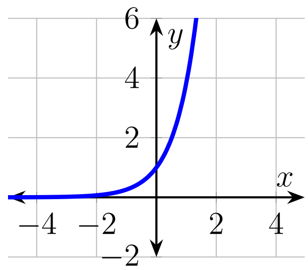
14. Match each function with its corresponding graph. (Page 223) (Solution 14)

$f(x) = 1(1+.4)^{((1/4)x)}$ $\quad f(x) = 1(1+.4)^{(4x)}$ $\quad f(x) = 1(1+.4)^{x}$

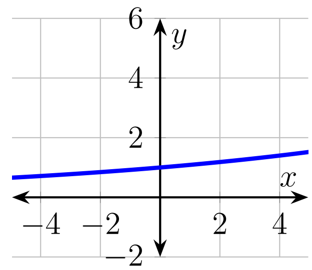
Graph A



Graph B



Graph C



15. For the logarithmic expression $\log_2 16 = 4$, identify the base, argument, and logarithm. (Page 225) (Solution 15)

16. For the logarithmic expression $\ln 20.086 = 3$, identify the base, argument, and logarithm. (Page 225) (Solution 16)

17. For the logarithmic expression $\log 1000 = 3$, identify the base, argument, and logarithm. (Page 225) (Solution 17)

18. Evaluate $\log_3 81$. (Page 225) (Solution 18)

19. Evaluate $\log 10,000$. (Page 225) (Solution 19)

20. Logarithmic Laws - True or False: $\log_4(5(7)) = \log_4 5 + \log_4 7$
If False, then correct the expression. (Page 225) (Solution 20)

21. Logarithmic Laws - True or False: $\log_3 \frac{4}{7} = \log_3 4 - \log_3 11$
If False, then correct the expression. (Page 225) (Solution 21)

22. Logarithmic Laws - True or False: $\log_3(3^6) = 6\log_3 4$
If False, then correct the expression. (Page 225) (Solution 22)

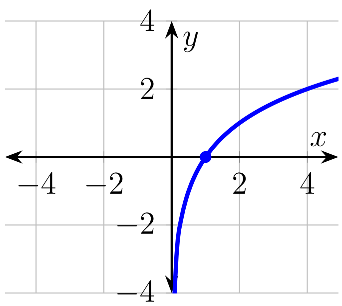
23. Logarithmic Laws - True or False: $\log_{10} 10^7 = 8\log_{10} 10$
If False, then correct the expression. (Page 225) (Solution 23)

24. Logarithmic Laws - True or False: $\log_{10}\sqrt[2]{100} = \frac{1}{2}\log_{10} 50$
If False, then correct the expression. (Page 225) (Solution 24)

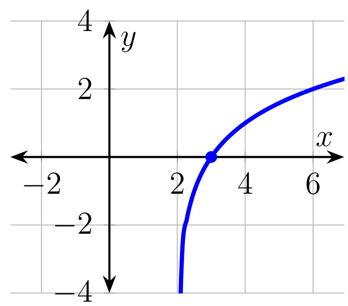
25. Logarithmic Laws - True or False: $\log_{2}\sqrt[2]{16} = \frac{1}{2}\log_{2} 2,$
If False, then correct the expression. (Page 225) (Solution 25)

26. True or False: $\ln 1 = 3$ If False, then correct the expression.
(Page 227) (Solution 26)

27. The function $f(x) = \log_2 x$ is graphed below. Identify the domain and range.
(Page 230) (Solution 27)

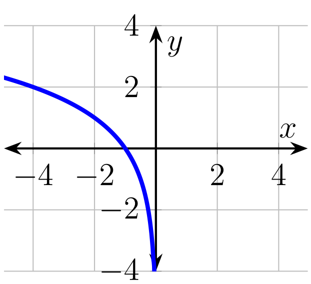


28. The function $f(x) = \log_2(x-2)$ is graphed below. Identify the domain and range.
(Page 230) (Solution 28)

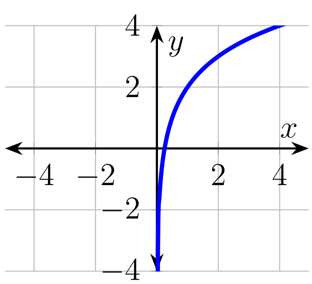


29. Three graphs below are based on the function $f(x) = \log_2 x$. Match the functions listed here with the corresponding graphs below. (Page 231) (Solution 29)
$f(x) = \log_2(x) + 2$ $\quad f(x) = \log_2(x-2)$ $\quad f(x) = \log_2(-x).$

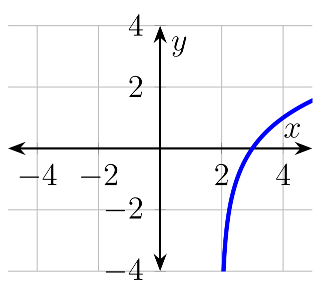
Graph A



Graph B

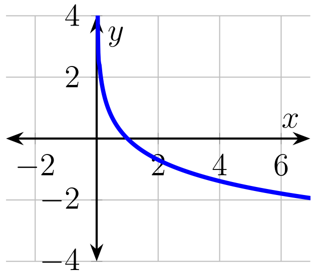


Graph C

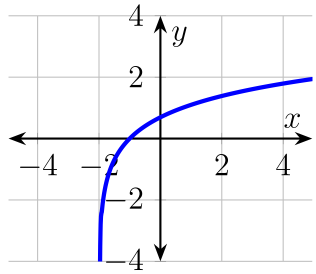


30. Three graphs below are based on the function $f(x) = \ln x$. Match the functions listed here with the corresponding graphs below. (Page 231) (Solution 30)
$f(x) = \ln(x) - 2$ $\quad f(x) = \ln(x+2)$ $\quad f(x) = -\ln x.$

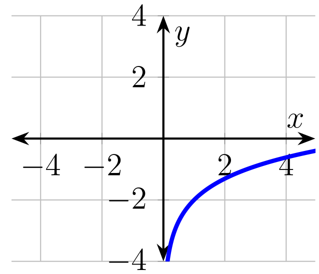
Graph D



Graph E



Graph F



7.8 Exponential & Logarithmic Functions Solutions

1. Graph A
2. Graph E
3. Graph I
4. Initial value: 3 Growth rate: .2
5. Initial value: (1/2) Decay rate: -.3
6. Initial value: .75 Decay rate: -.6
7. Decay
8. Growth
9. Decay
10. Growth
11. Graph A is $f(x) = 2^x$, Graph B is $f(x) = 2^{(x-2)}$, Graph C is $f(x) = 2^x - 2$
12. Graph D is $f(x) = -2^{(x+2)}$, Graph E is $f(x) = -2^x$, Graph F is $f(x) = 2^x$
13. Graph G is $f(x) = 4(1+.4)^x$, Graph H is $f(x) = (1+.4)^x$, Graph I is $f(x) = (1/2)(1+.4)^x$
14. Graph A is $f(x) = (1+.4)^x$, Graph B is $f(x) = 1(1+.4)^{(4x)}$, Graph C is $f(x) = 1(1+.4)^{((1/4)x)}$
15. Base: 2, Argument: 16, Logarithm: 4
16. Base: e, Argument: 20.086, Logarithm: 3
17. Base: 10, Argument: 1000, Logarithm: 3
18. 4 because $3^4 = 81$
19. 4 because $10^4 = 10,000$
20. True
21. False, $\log_3 \frac{4}{7} = \log_3 4 - \log_3 7$
22. False, $\log_3(3^6) = 6\log_3 3$
23. False, $\log_{10} 10^7 = 7\log_{10} 10$
24. False, $\log_{10} \sqrt[2]{100} = \frac{1}{2}\log_{10} 100$

25. False, $\log_2 \sqrt[2]{16} = \dfrac{1}{2}\log_2 16$

26. False, $\ln 1 = 0$ or $\log_e 1 = 0$

27. Domain: $(0, \infty)$, $0 < x$, or $\{x \mid 0 < x\}$
Range: all real numbers, $(-\infty, \infty)$, or $\{y \mid y \in \mathbb{R}\}$

28. Domain: $(2, -\infty)$, $2 < x$, or $\{x \mid 2 < x\}$
Range: all real numbers, $(-\infty, \infty)$, or $\{y \mid y \in \mathbb{R}\}$

29. Graph A is $f(x) = \log_2(-x)$, Graph B is $f(x) = \log_2(x) + 2$, Graph C is $f(x) = \log_2(x-2)$

30. Graph D is $f(x) = -\ln x$, Graph E is $f(x) = \ln(x+2)$, Graph F is $f(x) = \ln(x) - 2$

Chapter 8: Absolute Value Functions

OVERVIEW

The sections of this chapter are:

8.1 Absolute Value Equation
8.2 Absolute Value Inequalities
8.3 Visual Absolute Value Functions
8.4 Absolute Value Function Transformations

Measurement always relies on a reference point. Parents mark a child's growth on a door frame, tracking height over time. Daily temperatures fluctuate between hot and cold. Whether measuring height or temperature, a reference point is essential, based on local or international standards.

Measurement hinges on finding the difference between a value and a reference point, akin to absolute value. Simply instructing a friend to travel 5 yards or meters lacks context. Establishing a starting point, like "5 yards forwards from the front door," adds clarity. Absolute value is just the difference between a value and zero. Whether measuring a puppy's height, a kitten's weight, or reading a thermometer, understanding absolute values is straightforward.

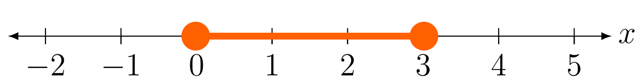
OBJECTIVES

By the end of the chapter, a student will be able to:

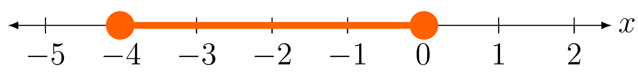
- Discuss and understand absolute values
- Solve absolute value equations
- Visualize, recognize, and interpret absolute value function graphs
- Translate, scale, and reflect absolute value function graphs

8.1 Absolute Value Equations

An absolute value is the difference between a value and zero. The symbols regarding absolute are just another way of communicating a simple concept. An interval is simply a difference between two values. Likewise, an absolute value is the difference between a value and zero. For the real, whole number 3, this is shown below, on a number line.



In the number line, above, the difference between 3 and zero is simply 3. In other words, the absolute value of 3 is 3. The concept of absolute value is described with the absolute value symbol | |. So $|3| = 3$.



The number line above demonstrates $|-4| = 4$, which means that the absolute value of -4 is equal to 4. For a negative number, the absolute value operation removes the negative, so that the output can only be a positive number. The input and output nature of an absolute value operation is demonstrated below.

Fig. 8.1: Absolute Value

$$\textbf{input} = -5 \longrightarrow \boxed{|input|} \longrightarrow \textbf{output} = \mathbf{5}$$

Definition 8.1 — Absolute Value. Absolute value describes the difference between a value and zero. This difference will not include a direction, so the absolute value is always positive. The absolute value symbol | | surrounds the input variable x.

$$|x| = \begin{cases} x & 0 \leq x \\ -x & x < 0 \end{cases} \qquad \begin{array}{l} \text{If } x \text{ is positive, then the output is } x \\ \text{If } x \text{ is negative, negate it to make it positive} \end{array}$$

In simple terms, any negative is removed so that the output is positive. The absolute value of x can be written as $|x|$ or $abs(x)$.

Solved Problem 8.1 Solve the following expressions, for y.

(a) $|-26| = y$ (b) $|7| = y$ (c) $-|78| = y$ (d) $|-125| = y$

(a) $|-26| = \boxed{26 = y}$ (b) $|7| = \boxed{7 = y}$

(c) $-|78| = \boxed{-78 = y}$ (d) $|-125| = \boxed{125 = y}$

Absolute Value Laws

1. $|a| \geq 0$
2. $|a| = |-a|$
3. $|ab| = |a||b|$
4. $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$

Absolute value law 1 states that any absolute value is greater than or equal to zero. Law 2 indicates that the absolute value of a number and its negative are equal. Law 3 states that the absolute value of a product equals the product of the absolute values of the factors. Law

4 states that the absolute value of a ratio equals the absolute value of the numerator divided by the absolute value of the denominator. Law 2 implies two inputs can yield one output, as illustrated below

$$|-5| = 5 \text{ and } |5| = 5 \qquad |-8| = 8 \text{ and } |8| = 8 \qquad \text{if } |x| = 9 \text{ then } x = 9 \text{ or } x = -9$$

For this reason, solving an absolute value equation requires solving two equations. The following steps will be useful while working with absolute value equations.

1. Isolate the absolute value.
2. Is the absolute value equal to a negative value?
3. Write and solve 2 equations.

Notice that step 2, just above, checks if the absolute value is equal to a negative value. If this occurs, the process is over, and there is no solution to the absolute value equation. This is because absolute value law 1, above, states that an absolute value cannot be equal to a negative value.

Step 1. Isolate the absolute value.

If $|x-5| = 4$ the absolute value is already isolated.

Step 2. Is the absolute value equal to a negative value?

Also, this absolute value is not equal to a negative value.

Step 3. Write and solve 2 equations.

In the next step, two equations are written. One equation is set equal to 4, and the second equation is set equal to -4. This will lead to two possible solutions.

$$x - 5 = 4 \quad \text{then } x - 5 \boxed{+5} = 4 \boxed{+5} \qquad \boxed{x = 9}$$

$$x - 5 = -4 \quad \text{then } x - 5 \boxed{+5} = -4 \boxed{+5} \qquad \boxed{x = 1}$$

Solved Problem 8.2 Solve the absolute value equation $|3x-6| - 9 = -3$.

Step 1. Isolate the absolute value

$$\text{If } |3x-6| - 9 = -3 \qquad |3x-6| - 9 \boxed{+9} = -3 \boxed{+9} \qquad |3x-6| = 6$$

Step 2. Is the absolute value equal to a negative value? Now that the absolute value has been isolated, one can see that it is not equal to a negative value.

Step 3. Write and solve 2 equations.

$3x - 6 = 6$ then $3x - 6 \boxed{+6} = 6 \boxed{+6}$ then $3x = 12$ $\boxed{x = 4}$

$3x - 6 = -6$ then $3x - 6 \boxed{+6} = -6 \boxed{+6}$ then $3x = 0$ $\boxed{x = 0}$ ■

8.2 Absolute Value Inequalities

Working with absolute value inequalities will be similar to working with absolute value equations. It is now clear that an absolute value cannot be equal to a negative number. The following steps will be useful while working with absolute value inequalities

Step 1: Isolate the absolute value, then write 2 equations.
Step 2: The first equation simply drops the absolute value symbol.
Step 3: The second equation reverses the inequality and negates the value.

Step 1. Isolate the absolute value, then write 2 equations.

If $|x - 1| < 2$ the absolute value is already isolated.

Step 2. The first equation simply drops the absolute value symbol.

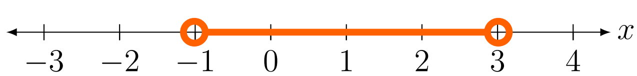
$x - 1 < 2$ then $x - 1 \boxed{+1} < 2 \boxed{+1}$ $\boxed{x < 3}$

One can see in Step 2, above, and Step 3, below that 1 is added to both sides of the equation to isolate the variable x.

Step 3. The second equation reverses the inequality and negates the value.

$x - 1 > -2$ then $x - 1 \boxed{+1} > -2 \boxed{+1}$ $x > -1$ or $\boxed{-1 < x}$

These solutions $-1 < x$ and $x < 3$ are demonstrated on the number line below.



A compound inequality makes use of two inequalities. Each of the two inequalities is handled based on the concepts above. One identifies each of the two inequalities and proceeds with the same steps listed above. This is demonstrated below.

$1 < |x + 2| < 3$ consists of 2 inequalities $\boxed{1 < |x + 2|}$ and $\boxed{|x + 2| < 3}$. Each of these is first handled separately

Step 1. Isolate the absolute value.

If $\boxed{1 < |x + 2|}$ the absolute value is already isolated.

Step 2. The first equation simply drops the absolute value symbol.

$1 < x+2$ then $1 \boxed{\text{-2}} < x+2 \boxed{\text{-2}}$ $\boxed{-1 < x}$

One can see in Step 2, above, and Step 3, below that 2 is subtracted from both sides of the equation to isolate the variable x.

Step 3. The second equation reverses the inequality and negates the value.

$-1 > x+2$ then $-1 \boxed{\text{-2}} > x+2 \boxed{\text{-2}}$ $-3 > x$ or $\boxed{x < -3}$

These same 3 steps, just above are now applied to the second inequality of the compound inequality.

Step 1. Isolate the absolute value.

If $\boxed{|x+2| < 3}$ the absolute value is already isolated.

Step 2. The first equation simply drops the absolute value symbol.

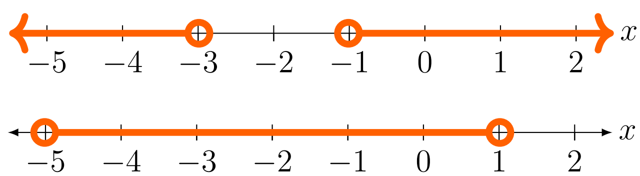
$x+2 < 3$ then $x+2 \boxed{\text{-2}} < 3 \boxed{\text{-2}}$ $\boxed{x < 1}$

One can see in Step 2, above, and Step 3, below that 2 is subtracted from both sides of the equation to isolate the variable x.

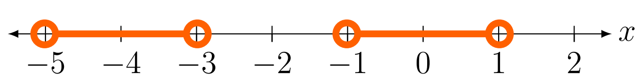
Step 3. The second equation reverses the inequality and negates the value.

$x+2 > -3$ then $x+2 \boxed{\text{-2}} > -3 \boxed{\text{-2}}$ $x > -5$ or $\boxed{-5 < x}$

The first two solutions $x < -3$ and $-1 > x$ are demonstrated on the first number line below. Then the solutions $-5 < x$ and $x < 1$ are demonstrated on the second number line below.



These two number lines are now used to arrive at the final solution of the compound inequality $1 < |x+2| < 3$. This is done by finding the overlap or intersection of the two number lines, just above. This is shown below.



This solution is $\boxed{(-5,-3) \text{ and } (-1,1)}$, or $\boxed{\{x \mid -5 < x < -3,\ -1 < x < 1\}}$.

The absolute value laws can be quite a time saver when working with absolute value inequalities. The absolute value laws were listed on page 240.

Solved Problem 8.3 Solve the absolute value inequalities

(a) $|5x-3| \leq -3$ (b) $|17x+2| \leq -4$

(a) According to absolute value law 1, $|a| \geq 0$. In other words an absolute value must be greater than or equal to zero.

For $|5x-3| \leq -3$ there is $\boxed{\text{No solution}}$

(b) Again, an absolute value must be greater than or equal to zero.

For $|17x+2| \leq -4$ there is $\boxed{\text{No solution}}$

Solved Problem 8.4 Solve the absolute value inequality $|5-3x| > 8$

Step 1. Isolate the absolute value, then write 2 equations.

If $|5-3x| > 8$ the absolute value is already isolated.

Step 2. The first equation simply drops the absolute value symbol.

$$5-3x > 8 \quad \text{then } 5-3x \boxed{-5} > 8 \boxed{-5} \qquad \frac{\cancel{-3}x}{\cancel{-3}} > \frac{3}{-3} \qquad \boxed{x < -1}$$

Step 3. The second equation reverses the inequality and negates the value.

$$5-3x < -8 \quad \text{then } 5-3x \boxed{-5} < -8 \boxed{-5} \qquad \frac{\cancel{-3}x}{\cancel{-3}} > \frac{13}{-3} \qquad \boxed{x > -\frac{13}{3}}$$

$$\boxed{-\frac{13}{3} < x < -1 \quad \text{or} \quad \left(-\frac{13}{3}, -1\right) \quad \text{or} \quad \left\{x \mid -\frac{13}{3} < x < -1\right\}}$$

8.3 Visual Absolute Value Functions

What does it mean to describe a function with clarity and specifics? How would one describe and discuss an absolute value function with regard to the following concepts?

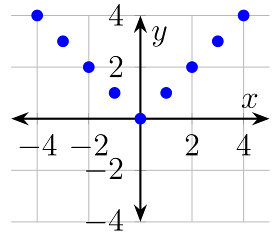
- Slope
- Domain
- Range
- X-intercept
- Y-intercept
- One-to-one mapping
- Nonlinearity
- Symmetry
- Odd or Even

Recall that an initial approach to begin to understand a function is to begin plotting some points. This can also be described as reviewing the input, output nature of the function.

Each coordinate pair, includes an input and output.

If $x = -4$, $y = 4$
If $x = -3$, $y = 3$
If $x = -2$, $y = 2$
If $x = -1$, $y = 1$
If $x = 0$, $y = 0$
If $x = 1$, $y = 1$
If $x = 2$, $y = 2$
If $x = 3$, $y = 3$
If $x = 4$, $y = 4$

Fig. 8.2: $f(x) = |x|$



The absolute value function graph is nonlinear, due to the corner at (0, 0). It does consist of two pieces or components that are linear, or straight lines.

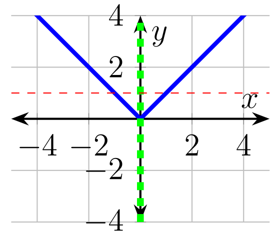
All outputs are greater than or equal to zero.

The green, dashed line is an axis of symmetry.

It is an even function.

It is nonlinear.

Fig. 8.3: $f(x) = |x|$



The domain of an absolute function is all real numbers, or $(-\infty, +\infty)$. For the positive domain of $[0, +\infty)$, the slope is 1. Then, for the negative domain $(-\infty, 0]$, the slope is -1. The range then is greater than or equal to zero, or $[0, +\infty)$. Also, an even function is defined by the condition $f(x) = f(-x)$. Testing for this condition is shown below.

$$f(x) = \boxed{|x|} \text{ and } f(-x) = |-x| = |-1||x| = 1|x| = \boxed{|x|} \text{ so } f(x) = f(-x)$$

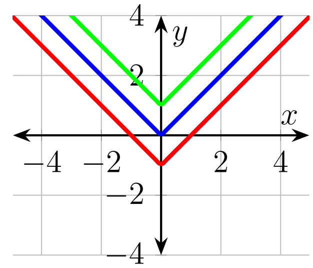
The horizontal line test confirms that this function does not display **one-to-one mapping**. In the graph above, the red, dashed horizontal line test at $y = 1$, shows that both $x = -2$ and $x = 2$ produce the same output of $y = 2$. One can see that the red, dashed, horizontal line touches the graph at 2 points at a time, which does not occur in one-to-one mapping.

8.4 Absolute Value Function Transformations

Recognition of the absolute value function graph can be reinforced by applying transformations to it.

The graph below illustrates vertical translation. Vertical translation refers to either shifting up or down. A constant c, is added to or subtracted from the function. Again, the y-intercept responds to either a shift up or down.

Fig. 8.4: $f(x) = |x| + c$

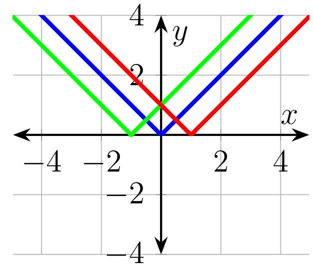


The blue graph shows $f(x) = |x|$ or $abs(x)$

The green graph shows $f(x) = |x| + 1$. The constant 1, is added. The curve shifts up.

The red graph shows $f(x) = |x| - 1$. The constant 1, is subtracted. The curve shifts down.

Fig. 8.5: $f(x) = |x + c|$



The blue graph shows $f(x) = |x|$ or $abs(x)$

The green graph shows $f(x) = |x + 1|$. The constant 1, is added to the variable. The curve shifts to the left.

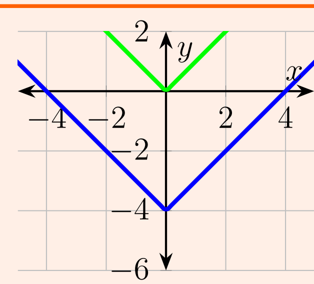
The red graph shows $f(x) = |x - 1|$. The constant 1, is subtracted from the variable. The curve shifts to the right.

Again, in the graph above, the constant c can be negative, as in -1. Then $f(x) = |x - 1|$, involves subtraction and a horizontal shift to the right.

Solved Problem 8.5 Graph the function $f(x) = |x| - 4$.

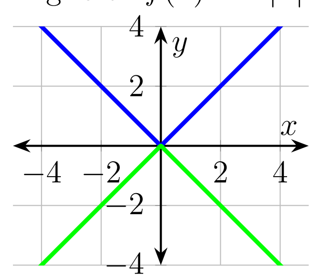
The function $f(x) = |x|$ is a starting point and this is shown in green.

The term $\boxed{-4}$ simply causes the function to shift down 4 units.



The graph below illustrates reflection, about the x-axis. A constant -1, is multiplied by the whole function.

Fig. 8.6: $f(x) = -|x|$



The blue graph shows $f(x) = |x|$

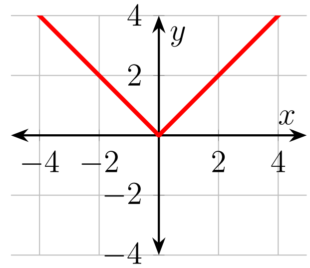
The green graph shows $f(x) = -|x|$. The whole function is negated. The curve reflects about the x-axis.

The graph below illustrates reflection about the y-axis, applied to the Absolute function $f(x) = |-x|$. A constant -1, is multiplied by the variable x, within. Since this is an even function, the function appears unchanged.

The blue graph would show $f(x) = |x|$

The red graph shows $f(x) = |-x|$. Only the variable within, is negated. The curve shifts about the y-axis. This red curve would overlap the blue curve. The location of the blue curve would not change.

Fig. 8.7: $f(x) = |-x|$

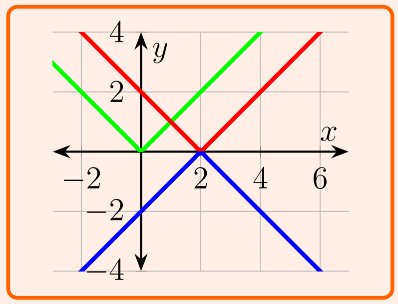


Solved Problem 8.6 Graph the function $f(x) = -|x-2|$.

The function $f(x) = |x|$ is a starting point and this is shown in green.

The function $f(x) = |x-2|$ is shift to the right, and this is shown in red.

The function $f(x) = -|x-2|$ reflects about the y-axis, upside down. This is shown in blue



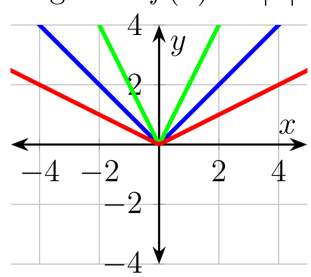
The graph below illustrates vertical scaling. Vertical scaling refers to either vertical stretching or vertical compression. The whole function is multiplied by a constant c, that can also be called a vertical scaling factor.

The blue graph shows $f(x) = |x|$

The green graph shows $f(x) = 2|x|$. The constant c, is the whole number 2, vertical stretching occurs.

The red graph shows $f(x) = (1/2)|x|$. The constant c, is the fraction (1/2), vertical compression occurs.

Fig. 8.8: $f(x) = c|x|$



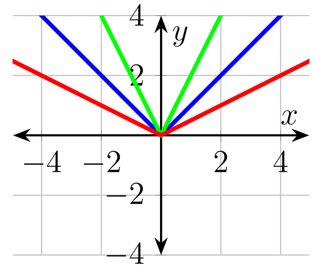
The graph below illustrates horizontal scaling. Horizontal scaling refers to either horizontal stretching or horizontal compression. For the graph below, the variable x is multiplied by a constant c, that can also be called a horizontal scaling factor.

The blue graph shows $f(x) = |x|$

The green graph shows $f(x) = |2x|$. The constant c, is the whole number 2, horizontal compression occurs.

The red graph shows $f(x) = |(1/2)x|$. The constant c, is the fraction $(1/2)$, horizontal stretching occurs.

Fig. 8.9: $f(x) = |cx|$

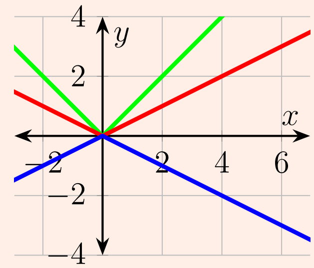


Solved Problem 8.7 Graph the function $f(x) = -|(1/2)x|$.

The function $f(x) = |x|$ is a starting point and this is shown in green.

The function $f(x) = |(1/2)x|$ has a factor within the function. This horizontal stretch is shown in red.

The function $f(x) = -|(1/2)x|$ reflects about the y-axis, upside down. This is shown in blue



8.5 Absolute Value Function Problems

Click on the page number to the right to see the chapter discussion. Click on the solution number to navigate to the solution.

1. Solve the following expressions, for y. (a) $|-520| = y$ (b) $|39| = y$
   (Page 240) (Solution 1)

2. Solve the following expressions, for y. (a) $|-152| = y$ (b) $|-78| = y$
   (Page 240) (Solution 2)

3. $f(x) = 2|x| - 10$ find $f(-10)$
   (Page 240) (Solution 3)

4. $f(x) = -3|x| + 8$ find $f(-5)$
   (Page 240) (Solution 4)

5. True or False - Absolute value laws. If False, then correct the expression.

   (a) $|a| = -|a|$ (b)$|a| > 0$ (Page 240) (Solution 5)

6. True or False - Absolute value laws. If False, then correct the expression.

   (a) $|ab| = -|a||b|$ (b)$\left|\frac{a}{b}\right| = -\frac{|a|}{|b|}$ (Page 240) (Solution 6)

7. Solve the following expressions, for x. (a) $|2x-1| + 5 = 14$ (b) $|-2x+7| = 25$

(Page 240) (Solution 7)

8. Solve the following expressions, for x. (a) $|2x+10|=0$ (b) $|3x-5|=0$

   (Page 240) (Solution 8)

9. Solve the following expressions, for x. (a) $|-x+7|=20$ (b) $|3x-6|-9=6$

   (Page 240) (Solution 9)

10. Solve $|10x+2|>0$
    (Page 244) (Solution 10)

11. Solve $|9x-3|>0$
    (Page 244) (Solution 11)

12. Solve $|3x+5|<7$
    (Page 244) (Solution 12)

13. Solve $|3x-2|-4<7$
    (Page 244) (Solution 13)

14. Solve $|3x+9|<18$
    (Page 244) (Solution 14)

15. Solve $|8x+4|\geq 0$
    (Page 244) (Solution 15)

16. Solve $|15x+5|\geq 0$
    (Page 244) (Solution 16)

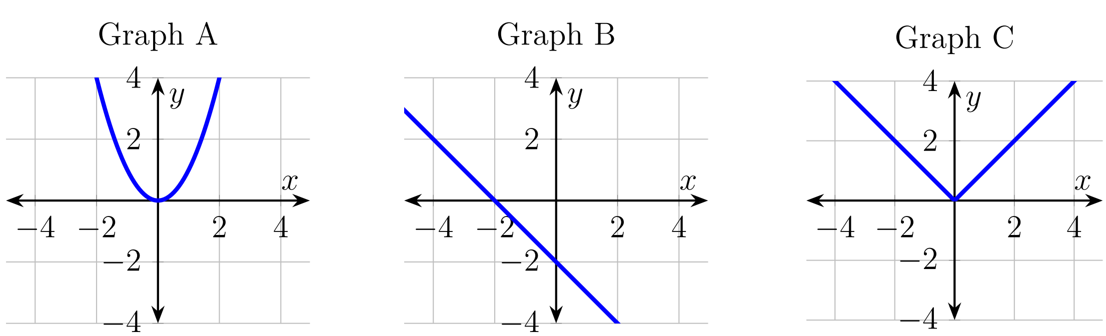
17. Solve $|13x+1|\leq -9$
    (Page 244) (Solution 17)

18. Solve $|144x-50|\leq -12$
    (Page 244) (Solution 18)

19. Solve $5-3|x-5|\geq -4$
    (Page 244) (Solution 19)

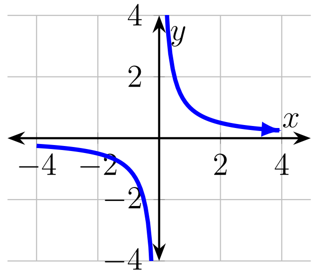
20. Solve $|13-3x|\geq 4$ (Page 244) (Solution 20)

21. Which graph below displays an absolute value function? (Page 244) (Solution 21)

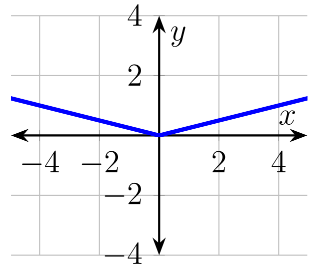


22. Which graph below displays an absolute value function? (Page 244) (Solution 22)

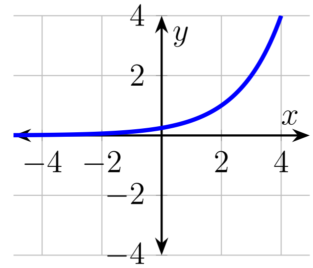
Graph D



Graph E



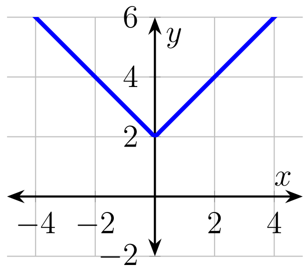
Graph F



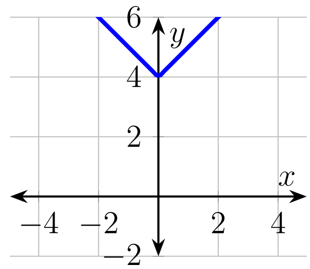
23. True or False: The function $f(x) = |x|$ displays one-to-one mapping.
(Page 245) (Solution 23)

24. Match the functions listed here with the corresponding graphs below.
$f(x) = |x| + 2$ $f(x) = |x|$ $f(x) = |x| + 4$. (Page 245) (Solution 24)

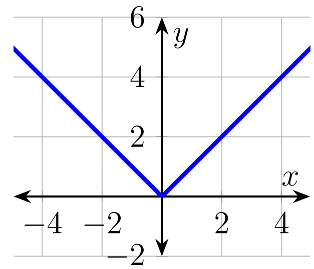
Graph G



Graph H

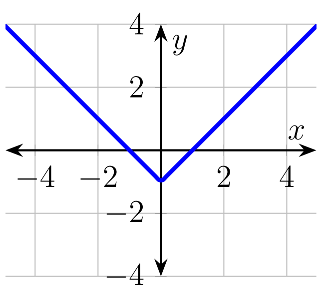


Graph I

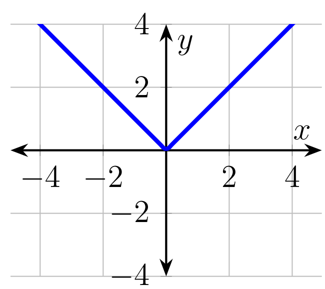


25. Match the functions listed here with the corresponding graphs below.
$f(x) = |x|$ $f(x) = |x| - 3$ $f(x) = |x| - 1$. (Page 245) (Solution 25)

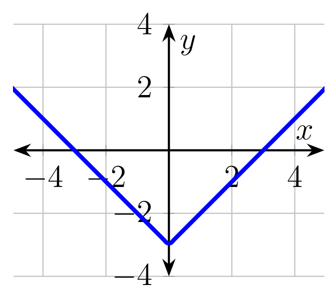
Graph A



Graph B



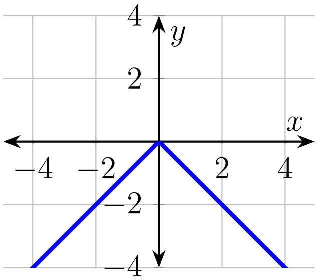
Graph C



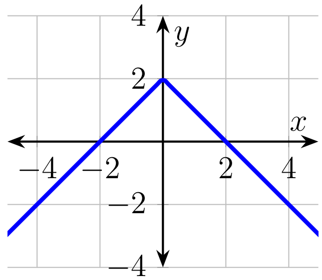
26. Match the functions listed here with the corresponding graphs below.

$f(x) = -|x| + 2 \qquad f(x) = |x| \qquad f(x) = -|x|.$ (Page 246) (Solution 26)

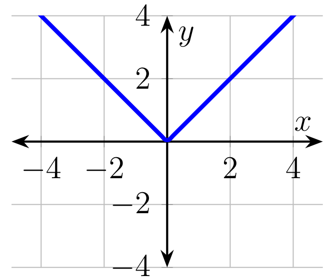
Graph D



Graph E



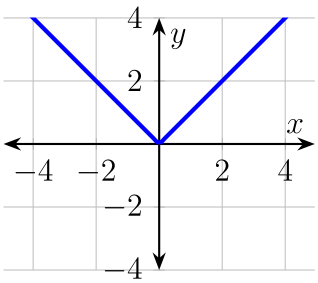
Graph F



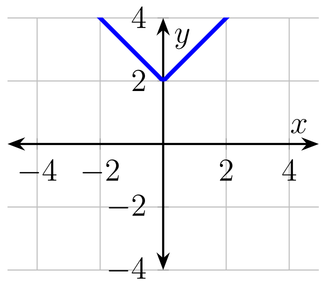
27. Match the functions listed here with the corresponding graphs below.

$f(x) = |-x| - 2 \qquad f(x) = |-x| + 2 \qquad f(x) = |x|.$ (Page 245) (Solution 27)

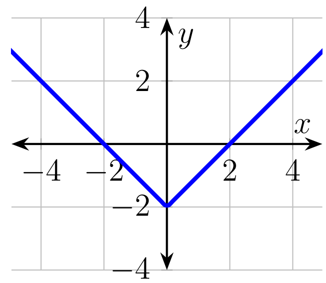
Graph G



Graph H



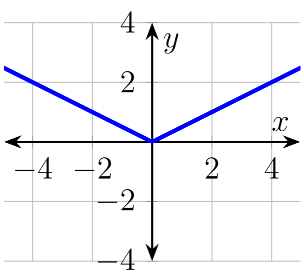
Graph I



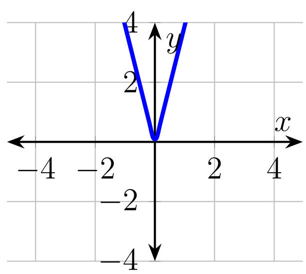
28. Match the functions listed here with the corresponding graphs below.

$f(x) = 4|x| \qquad f(x) = |x| \qquad f(x) = (1/2)|x|.$ (Page 245) (Solution 28)

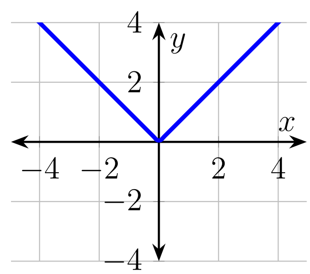
Graph A



Graph B



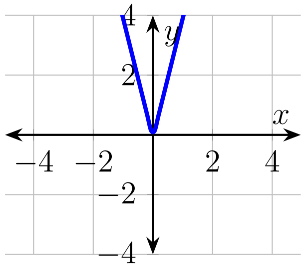
Graph C



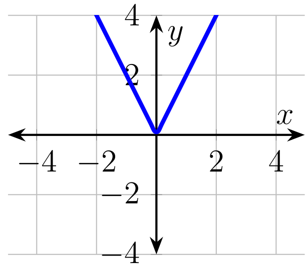
29. Match the functions listed here with the corresponding graphs below.

$f(x) = |2x| \qquad f(x) = |(1/3)x| \qquad f(x) = |4x|.$ (Page 245) (Solution 29)

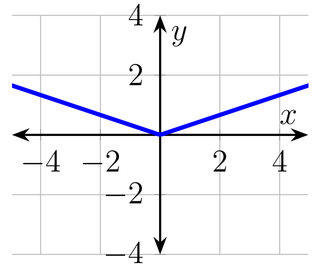
Graph D



Graph E

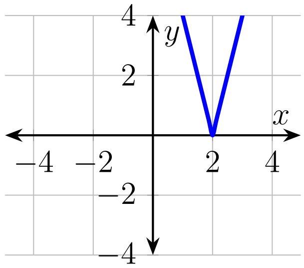


Graph F

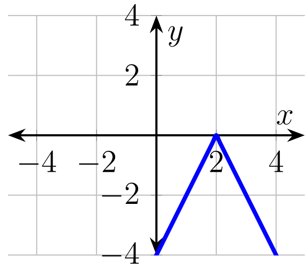


30. Match the functions listed here with the corresponding graphs below.
$f(x) = |(1/2)x| + 2$ $\quad f(x) = 4|x-2|$ $\quad f(x) = -2|x-2|$. (Page 245) (Solution 30)

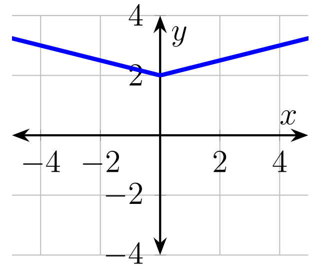
Graph G



Graph H



Graph I



8.6 Absolute Value Function Solutions

1. (a) $y = 520$ (b) $y = 39$
2. (a) $y = 152$ (b) $y = 78$
3. $f(-10) = 10$
4. $f(-5) = -7$
5. (a) False, $|a| = |-a|$ (b) False, $|a| \geq 0$
6. (a) False, $|ab| = |a||b|$ (b) False, $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$
7. (a) $x = -4, 5$ (b) $x = -9, 16$
8. (a) $x = -5$ (b) $x = \dfrac{5}{3}$
9. (a) $x = -13, 27$ (b) $x = -3, 7$
10. $\left(-\infty, \dfrac{1}{5}\right) \cup \left(\dfrac{1}{5}, +\infty\right)$ or $\left\{x \mid x \neq \dfrac{1}{5}\right\}$
11. $\left(-\infty, \dfrac{1}{3}\right) \cup \left(\dfrac{1}{3}, +\infty\right)$ or $\left\{x \mid x \neq \dfrac{1}{3}\right\}$
12. $\left(-4, \dfrac{2}{3}\right)$ or $\left\{x \mid -4 < x < \dfrac{2}{3}\right\}$
13. $\left(-3, \dfrac{13}{3}\right)$ or $\left\{x \mid -3 < x < \dfrac{13}{3}\right\}$
14. $(-9, 3)$ or $\{x \mid -9 < x < 3\}$

15. $(-\infty,+\infty)$ or $\{x \mid x \in \mathbb{R}\}$

16. $(-\infty,+\infty)$ or $\{x \mid x \in \mathbb{R}\}$

17. No solution

18. No solution

19. $[2,8]$ or $\{x \mid 2 \le x \le 8\}$

20. $x \le 3$ and $x \ge \dfrac{17}{3}$ or $(-\infty,3] \cup \left[\dfrac{17}{3},+\infty\right)$ or $\left\{x \mid x \le 3, \dfrac{17}{3} \le x\right\}$

21. Graph C

22. Graph E

23. False

24. Graph G is $f(x)=|x|+2$, Graph H is $f(x)=|x|+4$, Graph I is $f(x)=|x|$

25. Graph A is $f(x)=|x-2|$, Graph B is $f(x)=|x|$, Graph C is $f(x)=|x|-3$

26. Graph D is $f(x)=-|x|$, Graph E is $f(x)=-|x|+2$, Graph F is $f(x)=|x+2|$

27. Graph G is $f(x)=|x|$, Graph H is $f(x)=|-x|+2$, Graph I is $f(x)=|-x|-2$

28. Graph A is $f(x)=(1/2)|x|$, Graph B is $f(x)=4|x|$, Graph C is $f(x)=|x|$

29. Graph D is $f(x)=|4x|$, Graph E is $f(x)=|2x|$, Graph F is $f(x)=|(1/3)x|$

30. Graph G is $f(x)=4|x-2|$, Graph H is $f(x)=-2|x-2|$, Graph I is $f(x)=|(1/2)x|+2$

Chapter 9: Binomial Theorem

OVERVIEW

The sections of this chapter are:

9.1 The Fundamental Counting Principle
9.2 Factorials
9.3 Combinations & Permutations
9.4 Binomial Expansion

In life, many situations have uncertain outcomes, like flipping a coin or drawing a card. From riding a bike to finding parking, uncertainty is common. Unlike controlled experiments, everyday life is full of unknowns.

Complex systems, like weather or financial markets, consist of many dynamic components. Predicting outcomes, such as the winner of a championship, is uncertain but can be assessed with terms like "likely" or "unlikely." Understanding certainty and uncertainty is crucial across various fields like probability, statistics, and business, aiding in describing the likelihood of outcomes.

OBJECTIVES

By the end of the chapter, a student will be able to:

- Understand why "deterministic" means a lot of math is easy!
- Apply the fundamental counting principle.
- Calculate the factorial of a number.
- Discuss and understand permutations.
- Discuss and understand combinations.
- Apply binomial expansion.

9.1 The Fundamental Counting Principle

Intro to counting principles, factorials, permutations, and combinations can seem complex. But grasping concepts before diving into math is key. Understanding uncertainty is crucial. Terms like "algorithm" may sound complex, but they're familiar concepts.

Definition 9.1 — Algorithm. A process or set of rules, that is meant to produce a specific output or outcome.

When defining an algorithm, one might ponder if a factory process qualifies. Indeed, a factory process is essentially an algorithm. The concept extends beyond math and computers, revolving around a process or set of rules. Even making a peanut butter and jelly sandwich is an example of an algorithm.

Contrasting uncertainty with deterministic algorithms clarifies their meanings. An algorithm is a predictable process; $2+2$ always equals 4. Deterministic means the output is consistent. Baking a cake or toasting bread are deterministic processes; the result is always the same. Introducing this concept early simplifies math as repeatable steps.

> **Definition 9.2 — Deterministic.** This refers to a function, system, or process where the output is always the same and known for a specific input. The output or result is known with certainty. Randomness does not play a role in a deterministic process or system.

In contrast, a nondeterministic algorithm yields varied outcomes each time it's executed. Such processes are probabilistic, involving probabilities. For instance, predicting the card drawn from a shuffled deck or the exact future temperature is impossible due to their unpredictability.

The fundamental counting princicple has many practical applications. It is a time saver, and it keeps one from having to count possibilities, one by one. If a password has only two lower case letters from the English alphabet how many possible passwords are there?

Working with possibilities and probability, involves interpreting a word problem or description. This requires practice. For this particular example the following points are important.

- Each password will have two characters.
- Each of the two characters is a lowercase letter.
- For each character, there are 26 possible lowercase letters.
- To make a password, first one lowercase letter is chosen, and then a second lowercase letter is chosen.
- Choosing the first lowercase letter, can be considered one event.
- Choosing the second lowercase letter, can be considered a second event.

One could try to count all the possibilities, in an orderly way, one by one, but this would take a very long time. For example, one would start with the letter "a" for the first letter and then cycle through the lower case alphabet for the second letter. This would be begin as shown below.

"aa" "ab" "ac" "ad" "ae" "af" "ag" "ah" "ai"

"ba" "bb" "bc" "bd" "be" "bf" "bg" "bh" "bi"

Each possibility, shown above, will have two lower case letters. In order to count, one will have to cycle through the entire alphabet for both letters. How long will it take to count all the possibilities. One can see, right away, that counting each possibility will take a very long time. An alternative is to use the fundamental counting principle, as shown below.

$$26 \times 26 = 26^2 = \boxed{676}$$

Definition 9.3 — Fundamental Counting Principle. If one event has m possible outcomes, and if a second event has n possible outcomes, the total number of possible outcomes for these two events is equal to $m \times n$.

Solved Problem 9.1 How many possible passwords exist if a password has two characters, where each character is a lowercase or an uppercase letter?

A first step is to interpret, in order to draw key information.

- There are two events.
- In the first event, an upper or lower case letter is chosen.
- In the second event, an upper or lower case letter is chosen.
- The sum of upper and lower case letters is 52.

In the first event, 52 possible letters can be chosen. $m = 52$

In the second event, 52 possible letters can be chosen. $n = 52$

Based on the fundamental counting principle, the total number of possible passwords is equal to $m \times n = 52 \times 52 = \boxed{2,704}$ ■

Solved Problem 9.2 A weekend conference will assign a unique ID with 2 characters to each person that attends the conference. Each unique ID will consist of 2 characters. The first character in each unique ID must a be number from 0 to 9. The second character must an uppercase letter. How many unique IDs will the event coordinator be able to assign?

- Each unique ID has 2 characters.
- In the first event one number from 0 to 9 will be chosen.
- In the second event an uppercase letter is chosen.

There are 10 possible digits as choices from 0 to 9, in the first event.

There are 26 possible letters as choices, for the second event.

Based on the fundamental counting principle, the total number of possible unique IDs is equal to $10 \times 26 = \boxed{260}$ ■

9.2 Factorials

In the solved problems, we've mentioned a total number of options. We can describe these options more precisely using the concept of factorials. Exponential notation shows how many times a number is multiplied by itself, like 2^4 is $2 \times 2 \times 2 \times 2$. A factorial expresses the multiplication of a number by all the numbers before it, down to 1.

Definition 9.4 — Factorial. The factorial of a non-negative number n is written as $n!$ and it is equal to the product of all the integers between 1 and n. This can can also be described in the following way.

$$!n = n \times (n-1) \times (n-2) \times ... \times 3 \times 2 \times 1$$

Examples of this are shown below.

$$3! = 3 \times 2 \times 1 \qquad\qquad 5! = 5 \times 4 \times 3 \times 2 \times 1$$

One can see in the examples, just above, that consecutive integers are being multiplied.

Solved Problem 9.3 Solve the following expressions: (a) 6! (b) 9!

(a) $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \boxed{720}$

(b) $9! = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \boxed{362{,}880}$ ■

Tip The factorial of zero is 1, $0! = 1$

One can see, in the last solved problem that 9! includes 6!. In other words $9! = 9 \times 8 \times 7 \times 6! = 9 \times 8 \times 7 \times 720$. Realizing this saves time while solving the last solved problem.

Solved Problem 9.4 Solve the following expressions:

(a) $3(3!) - 2$ (b) $\dfrac{4!}{2} + 2!$ (c) $3!\left(\dfrac{4!}{4} - 3!\right)$

(a) $3(3!) - 2 = 3\boxed{(3 \times 2 \times 1)} - 2 = 3(6) - 2 = \boxed{16}$

(b) $\dfrac{4!}{2} + 2! = \dfrac{\boxed{4 \times 3 \times 2 \times 1}}{2} + (2 \times 1) = \dfrac{24}{2} + (2) = 12 + 2 = \boxed{14}$

(c) $3!\left(\dfrac{4!}{4} - 3!\right) = (3 \times 2 \times 1)\left(\dfrac{24}{4} - 6\right) = 6\left(\dfrac{24}{4}\right) - 6(6) = 6(6) - 36 = \boxed{0}$ ■

9.3 Combinations & Permutations

The fundamental counting principle was discussed while considering possibilities and possible cases. The term combination can be used to describe such possibilities. A key point of working with combinations in algebra, probability, and statistics, is that for combinations order does not matter.

One can correctly say that a combination of 3 ingredients can be used to make a peanut butter and jelly sandwich. The combination could be described as shown below.

{bread, peanut butter, jelly}   {bread, jelly, peanut butter}

Tip Combinations do not take into account order. If different sets have different order, but they have the same exact elements, then they are all 1 combination.

Three ingredients can be combined to make a peanut butter and jelly sandwich. The order of the ingredients doesn't matter when considering combinations. So, if the ingredients are the same but in a different order, they are still considered the same combination.

The two sets above are the same combination but they are different permutations. Permutation is nearly synonymous, or nearly the same word as order. Clearly the two sets above, have a different order.

The difference between combinations and permutations is illustrated again below. Below, one can see three sets of letters.

{t, e, a}   {e, a, t}   {a, t, e}

Notice that these three sets of letters have the exact same letters. They are said to be equal combinations. This is because for combinations, order does not matter. The difference in these three sets is the order. For this reason, they are considered three different permutations. The difference between them is in the different meaning. In algebra, probability, and statistics it is important to understand that these three sets are the same combination, and they are three different permutations.

In everyday language, "combination" implies a specific order, like in a padlock code. But in algebra, "combination" refers to the selection of items regardless of order. So, 2, 12, 30 and 12, 30, 2 are considered the same combination in algebra, just different permutations.

Solved Problem 9.5 For each list of sets below, find the total number of combinations.

(a) {b, a, c}   {a, b, c}   {c, b, d}   {c, a, e}

(b) {3, 5, 7}   {5, 7, 2}   {2, 7, 9}   {9, 7, 2}

(a) $\boxed{\text{3 combinations}}$

This is because {b, a, c} and {a, b, c} are the same combination. Then {a, b, c}, {c, b, d}, and {c, a, e} are different combinations, because they have different letters, or elements.

(b) $\boxed{\text{3 combinations}}$

This is because {2, 7, 9} and {9, 7, 2} are the same combination. Then {3, 5, 7}, {5, 7, 2}, and {2, 7, 9} are different combinations. ■

For words in the English language, order of letters, is very important. Any word can be called a permutation of letters. In the English language, different permutations usually have a different meaning. For example, GAME and MAGE are two different permutations of the same letters. Again, GAME and MAGE are the same combination, of letters, because for combinations order does not matter.

Combinations and permutations are important in various fields like artificial intelligence, healthcare, and cybersecurity. In AI, understanding word order is vital for effective communication. In healthcare and biology, permutations of DNA bases determine life forms. In cybersecurity, permutations and combinations safeguard passwords and encryption, ensuring secure online interactions. Mathematical principles, like the Fundamental Counting Principle, aid in efficiently analyzing possibilities. Encryption relies on permutations to scramble information, highlighting their indispensable role in cybersecurity.

Tip Permutations take into account order. If sets have a different order or then they are different permutations.

Solved Problem 9.6 For the list of sets below, find the total number of permutations and combinations. {a, t, c} {c, a, t} {c, t, a} {a, t, c}

$\boxed{\text{3 permutations and 1 combination}}$ There are 3 permutations because 3 of these sets have the same letters, but different order. These 3 permutations are shown just below.

{a, t, c} {c, a, t} {c, t, a}

There is 1 combination because all four sets have the same letters ■

A previous solved problem, in Section 9.1, considered unique, two digit IDs for people attending a weekend conference. Consider a unique ID that consists of 2 digits. Each digit can vary between 0 and 9. One possible unique ID could be 24, and another possible unique ID could be 42. Both of these unique ID numbers consist of the digits 2 and 4, but the order is different. Clearly the ID number of 24 is different than the ID number of 42. This demonstrates how order can matter when one considers possibilities. Permutation goes hand-in-hand with the concept of order.

Throughout this chapter, contrast has been shown between permutations and combinations. Also, possibilities, factorials, and order have been thoroughly discussed.

Definition 9.5 — Permutations. Permutations refer to the total number of arrangements that are possible for a given set. Each permutation or possible arrangement will have a different order. One often needs to know the total number of permutations that is possible for a given set. Two formulas make it possible to solve for this. One formula, is for the case where elements can repeat. Another formula, is for the case where elements cannot repeat.

With repetition

$$P(n,r) = n^r$$

Without repetition

$$P(n,r) = \frac{n!}{(n-r)!}$$

In the formulas above, n is the total number of elements that one is choosing from. Then r is the total number of elements that are chosen.

To unlock the three-dial lock used for luggage, we're dealing with permutations since the order of the numbers matters. inserting 2, 3, and 5 is different from inserting 3, 5, and 2. Each dial can have any number from 0 to 9, including both ends of the range. Therefore, for each dial, there are 10 possibilities. Since repetition is allowed, we calculate the total number of permutations by multiplying the number of possibilities for each dial together.

The previous definition is applied to this challenge. One needs to find, or choose, three digits, to unlock this lock, so $r = 3$. Then each digit must be chosen from the set $[0,9]$. Recall that square brackets mean that 0 and 9 are included. Also, $n = 10$. Then, all that is needed is to apply the formula for permutations, with repetition. This is because this lock allows all three digits to be the same.

$$P(n,r) = n^r \quad P(10,3) = 10^3 = 10 \times 10 \times 10 = \boxed{1000} \text{ permutations}$$

Solved Problem 9.7 A simple lock, for luggage, or a suitcase, has 3 dials. This lock requires three digits to unlock. Each number can be chosen from the interval $[0,9]$. This lock does not allow any one digit to repeat. How many permutations exist?

This problem makes use of the permutation formula, without repetition.

Each digit is chosen from $[0,9]$, so $n = 10$. Three digits are needed so $r = 3$.

$$P(n,r) = \frac{n!}{(n-r)!} \qquad P(10,3) = \frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{3,628,800}{5,040} = \boxed{720}$$

■

Definition 9.6 — Combinations. Combinations refer to the total number of arrangements that is possible for a given set. For a combination order does not matter. Arrangements with the same elements, but different order, are considered the same combination. Two formulas make it possible to solve for the total number of combinations that is possible. One formula is for the case where elements can repeat. Another formula is for the case where elements cannot repeat.

With repetition

$$\binom{n+r-1}{r} = {}_nC_r = C(n,r) = \frac{(n+r-1)!}{r!(n-1)!}$$

Without repetition

$$\binom{n}{r} = {}_nC_r = C(n,r) = \frac{n!}{r!(n-r)!}$$

In the formulas above, n is the total number of elements that one is choosing from. Then r is the total number of elements that are chosen. $\binom{n}{r}$ is read as "n choose r", and it can also be written as ${}_nC_r$.

The notation $C(n,r)$ means combinations as a function of n and r. The input variables are then n and r. This means that "n choose r" is equal to the function of n and r, described above. The notation forms $\binom{n}{r}$, ${}_nC_r$, and $C(n,r)$ can all be used to describe combinations.

- The most common notation is $\binom{n}{r}$
- The notation ${}_nC_r$ is helpful with a calculator
- Artificial intelligence tools will often accept, as input, "n choose r", which means $\binom{n}{r}$

Again, one can see in the definition above that factorials, also contribute to combinations. A key step then is to understand if repetition is allowed or not.

Solved Problem 9.8 A team will be assembled that will consist of 3 team members. These team members will be chosen from a pool of 5 different individuals. What is the total number of combinations that are possible?

Key components of this word problem are individuals/humans, 3, and 5. Individuals cannot repeat, or the same individual cannot occur twice in one team. This will make use of the combinations formula without repetition.

The total number of elements n would be 5. The total number chosen r would be 3.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad \binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{120}{6(2)} = \frac{120}{12} = \boxed{10}$$

Solved Problem 9.9 Adam is preparing for a hiking trip. Adam needs to take 2 fruits with him. Adam can choose any combination from a pool of 6 fruits. In this pool, of fruits there are at least 2 of each fruit. What is the total number of possible combinations?

The word problem does not state that Adam cannot choose 3 of the same fruit, so repetition is allowed. Also, there are at least 3 of each kind of fruit. This means that it is possible to select the same fruit 3 times. This will make use of the combinations formula with repetition.

The total number of elements n would be 5. The total number chosen r would be 3.

$\binom{r+n-1}{r} = \dfrac{(r+n-1)!}{r!(n-1)!} \qquad \binom{r+n-1}{r} = \dfrac{(2+6-1)!}{2!(6-1)!} = \dfrac{7!}{2!5!} = \dfrac{5,040}{2(120)} = \dfrac{5,040}{240}\boxed{21}$ ■

Again, the first step, in the last solved problem is to understand whether repetition is allowed or not.

Tip For combinations without repetition $\binom{3}{0} = 1$ and $\binom{3}{3} = 1$.

9.4 Binomial Expansion

Solved Problem 9.10 Simplify the expressions: (a) $(x+y)^0$ (b) $(x+y)^1$

(a) $(x+y)^0 = \boxed{1}$ (b) $(x+y)^1 = \boxed{\text{x+y}}$ ■

The results of the last solved problem are summarized, below.

$(x+y)^0 = \boxed{1}$ Coefficient: 1

$(x+y)^1 = \boxed{\text{x+y}}$ Coefficients: 1 1

1

1 1

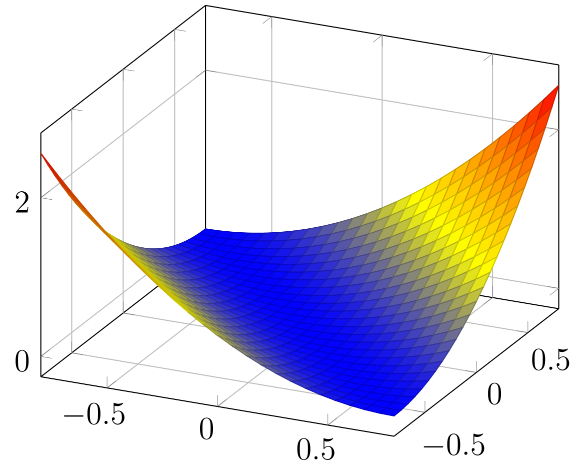
Solved Problem 9.11 Find the expansion of: (a) $(x+y)^2$ (b) $(x-y)^3$

(a) $(x+y)(x+y) \quad = x^2 + xy + xy + y^2 = \boxed{x^2 + 2xy + y^2}$

(b) $(x+y)^3 \;=\; \boxed{x^3 + 3x^2y + 3xy^2 + y^3}$

What would these results, or graphs look like? The second exercise (b) was solved by making use of Pascal's triangle. What is Pascal's triangle? ■

Fig. 9.1: $(x+y)^2$



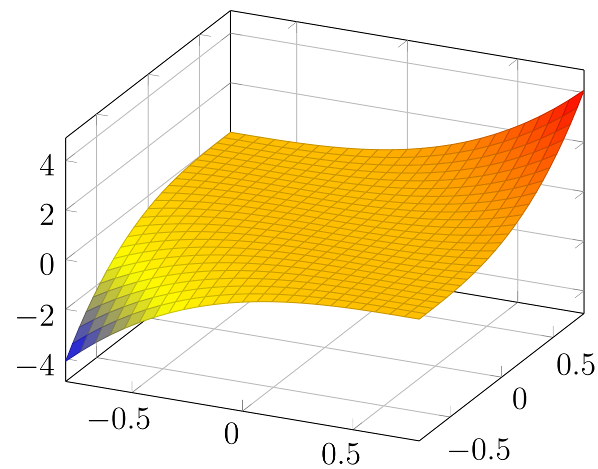
$(x+y)^2$ is shown here.

The domain is all real numbers.

The range or output is always greater than equal to zero.

Recall that exponents lead to curvy behavior.

Fig. 9.2: $(x+y)^3$



$(x+y)^3$ is shown here

The domain is all real numbers.

The range or output can be negative or positive.

Recall, that exponents lead to interesting curves. Just above, one can see the graphs of the two exercises in the last solved problem. As shown, in the last solved problem, binomial expansion leads to several terms and coefficients. These coefficients lead to an interesting pattern. The last two solved problems are summarized below. One should notice that a pattern is forming.

- The first number at the top is 1
- Each row increases by one additional number.
- Each row starts and ends with 1.
- Each number that is not at the start or end, is the sum of the two numbers directly above it.

One can separate the coefficients of binomial expansion. Below, to the right, one can see only the coefficients of binomial expansion. Notice, just below, that a triangle or pyramid shape is forming. Several observations can be made about this triangle.

$(x+y)^0 = \boxed{1}$ Coefficient: 1 $\qquad$ 1

$(x+y)^1 = \boxed{x+y}$ Coefficients: 1 1 $\qquad$ 1 1

$(x+y)^2 = \boxed{x^2+2xy+y^2}$ Coefficients: 1 2 1 $\qquad$ 1 2 1

Pascal's triangle follows a pattern where each number (except the first and last) in a row is the sum of the two numbers above it. The first row with only 1 is typically labeled row zero in zero-based counting, avoiding confusion with 1-based counting. Although some sources use 1-based counting, 0-based counting is more common. Pascal's triangle illustrates this pattern, with arrows showing the relationship between rows.

First row is called row 0

Row 1: Has the number 1 and 1

Row 2: 2 is the sum of 1 and 1

Row 3: 3 is the sum of 1 and 2

Row 4: 4 is the sum of 1 and 3

Row 4: 6 is the sum of 3 and 3

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

This is Pascal's triangle. This triangle can be used to find the coefficients of a binomial expansion. Reviewing this triangle, will confirm that there are several patterns.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

To expand $(x+y)^4$ quickly, use the fourth row of Pascal's triangle instead of manual binomial multiplication and the distributive law.

Tip In Pascal's triangle, the first row is usually called row zero.

Solved Problem 9.12 Given the following rows of Pascal's triangle, fill in the missing numbers.

1 6 15 20 15 6 1

1 $\boxed{}$ 21 $\boxed{}$ 35 $\boxed{}$ 7 1

1 $\boxed{}$ 28 $\boxed{}$ 70 $\boxed{}$ $\boxed{}$ 8 1

A clear pattern in Pascal's triangle is that one row makes it possible to find all the numbers in the next row. The first row is then used to find all the numbers in the second row.

The second row is filled-in with $1+6=7$, and $15+20=35$, and $6+1=7$. The third row is filled-in with

$$\begin{array}{ccccccccccccccccc}
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & \boxed{7} & & 21 & & \boxed{35} & & 35 & & \boxed{21} & & 7 & & 1 & \\
1 & & \boxed{8} & & 28 & & \boxed{56} & & 70 & & \boxed{56} & & \boxed{28} & & 8 & & 1
\end{array}$$

If zero based counting is used, row 5 of Pascal's triangle, provides the coefficients for the binomial expansion of $(x+y)^5$. Pascal's triangle is quite visual, but binomial expansion can be described in another way. The concepts will be the same, but it will be necessary to make use of a new symbol.

Definition 9.7 — Summation Notation. Summation notation describes repeated addition of terms. The summation symbol $\sum$ is based on the capital symbol sigma, from the greek alphabet. In this addition, the terms may also be called summands or addends. An index of addition describes the sequence of terms that is added. Summation notation makes it possible to describe a lower bound and/or an upper bound to the index of notation.

Symbol:	Examples:	Example:
Capital sigma $\sum$	$\sum_{i=2}^{i=4} i \quad \sum_{2}^{i=4} i \quad \sum_{2}^{4} i$	$\sum_{i=2}^{4} i = 2+3+4$

The letter i is the index of addition.

Tip In summation notation, i, j, k, n, or any letter can be used to show the index of addition.

$$\sum_{\boxed{\text{i=2}}}^{i=4} i = 2+3+4 \qquad \sum_{i=2}^{\boxed{\text{i=4}}} i = 2+3+4 \qquad \sum_{i=2}^{i=4} \boxed{i} = 2+3+4$$

The orange-bordered $i=2$ indicates the starting point for the sequence. In the middle, $i=4$ sets the endpoint. To the right, i in an orange border signifies the pattern or nature of each term. It's about summing integers from 2 to 4.

Solved Problem 9.13 Evaluate the following expressions

(a) $\sum_{i=1}^{i=4} i$ (b) $\sum_{i=5}^{i=10} i$ (c) $\sum_{i=1}^{i=3} i^2$

(a) $\sum_{i=1}^{i=4} i = 1+2+3+4 = \boxed{10}$ Addition of 1 through 4.

(b) $\sum_{i=7}^{i=10} i = 7+8+9+10 = \boxed{34}$ Addition of 7 through 10.

(c) $\sum_{i=1}^{i=3} i^2 = 1^2+2^2+3^2 = \boxed{14}$ Each term has a number that is squared. ■

Summation notation is demonstrated again below. The index of addition i has a lower bound of 2, and an upper bound of 4. One can see that in each term the constant 5 is added.

$$\sum_{i=2}^{i=4} i+5 \;=\; (2+5)+(3+5)+(4+5) = 7+8+9 = \boxed{24}$$

Solved Problem 9.14 Evaluate the following expressions

(a) $\sum_{i=1}^{i=3} i \times 2$ (b) $\sum_{i=2}^{i=4} i^2$ (c) $\sum_{i=1}^{i=3} (i!)$

(a) $\sum_{i=1}^{i=3} i \times 2 = (1\times 2)+(2\times 2)+(3\times 2) = 2+4+6 = \boxed{12}$
Each term requires multiplication by 2.

(b) $\sum_{i=2}^{i=4} i^2 = (2^2)+(3^2)+(4^2) = 4+9+16 = \boxed{29}$
Each term requires squaring the index of addition i.

(c) $\sum_{i=1}^{i=3} i! = (1!)+(2!)+(3!) = 1+2+6 = \boxed{9}$
Each term requires the factorial of the index of addition i. ■

Summation notation variations can include different representations of the upper and lower bounds. Here are more examples showcasing these variations.

$$\sum_{1}^{i=4} i \qquad \sum_{i=1}^{4} i \qquad \sum_{1}^{4} i \qquad \sum_{n=1}^{n=4} n \qquad \sum_{k=1}^{k=4} k$$

In these examples, either the lower or upper boundary is specified, not both. Additionally, either n or k can be used as the variable. The index of addition, usually denoted by i, may be replaced by any letter or even omitted. Now, with a grasp of summation notation and combinations, let's delve into understanding the for binomial expansion.

Definition 9.8 — Binomial Expansion/Theorem. The binomial theorem or binomial expansion describes the expanded form of a binomial to any exponent n. It makes it possible to describe this expanded form. It also makes it possible to solve for any one coefficient of the expanded form.

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i}\ b^i$$

$(a+b)$ is the binomial. Then n is the exponent that is applied to the binomial, and it is the upper limit in the summation notation.

The letter i is the index of addition. Then $\binom{n}{i}$ is $\dfrac{n!}{i!(n-i)!}$, or n choose i.

This formula shown, just above, for binomial expansion seems to have a lot of information in it. Again, thinking in terms of components can simplify a concept. Notice that key components of this formula are summation notation, combinations, and exponents.

Solved Problem 9.15 Evaluate the following expression $\quad (a+b)^3$

$$(a+b)^3 = \sum_{i=0}^{i=3} \binom{3}{i} a^{(3-i)} b^i \quad = \quad \binom{3}{0} a^{(3-0)} b^0 + \binom{3}{1} a^{(3-1)} b^1 + \binom{3}{2} a^{(3-2)} b^2 + \binom{3}{3} a^{(3-3)} b^3$$

$\binom{3}{0} = 1$ and $\binom{3}{3} = 1$. Then, one solves for the other two coefficients.

$$\binom{3}{1} = \frac{3!}{1!(3-1)!} = \frac{6}{2} = \boxed{3} \qquad \binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{6}{2} = \boxed{3}$$

Next, one needs to solve for the exponents, in each term. This is just subtraction, and this leads to the following binomial expansion.

$$a^{(3-0)} + 3a^{(3-1)} b^1 + 3a^{(3-2)} b^2 + a^{(3-3)} b^3 \quad = \quad \boxed{a^3 + 3a^2 b + 3ab^2 + b^3}$$

The binomial expansion formula can seem challenging, but notice that this last solved problem mostly depends on solving for only two coefficients. The rest is simpler and quicker. In this last solved problem, the coefficients match row 3 of Pascal's triangle.

9.5 Binomial Theorem Problems

Click on the page number to the right to see the chapter discussion. Click on the solution number to navigate to the solution.

1. If a restaurant has 5 beverages and 6 dishes, how many meals are possible?
   (Page 257) (Solution 1)

2. A travel agency can arrange travel to 3 different destinations. For each destination there 2 different airline travel packages. For each destination there are also 4 different hotel packages. How many different variations are possible at this travel agency?
   (Page 257) (Solution 2)

3. Adam buys 4 shirts and 3 pairs of shorts. If any and all the shirts match any and all of the pants. How may variations of shirts and shorts are possible?
   (Page 257) (Solution 3)

4. Solve the factorials
   (a) 4! (b) 7! (Page 258) (Solution 4)

5. Solve the factorials
   (a) 0! (b) 1! (Page 258) (Solution 5)

6. Solve the expressions (a) $\frac{6!}{2!}$ (b) $2!(4-2)!$ (Page 258) (Solution 6)

7. Solve the expressions (a) $\frac{5!}{3!}$ (b) $\frac{7!}{5!}$ (Page 258) (Solution 7)

8. How many combinations are there in the following list of sets?
   {50, 30, 10}, {40, 60, 50}, {30, 50, 10}, {10, 30, 50} (Page 259) (Solution 8)

9. How many combinations are there in the following list of sets?
   {A, R, K}, {R, K, A}, {K, P, A}, {W, C, A} (Page 259) (Solution 9)

10. How many permutations are there in the following list of sets?
    {A, R, K}, {R, K, A}, {A, K, R}, {A, R, K} (Page 260) (Solution 10)

11. How many permutations are there in the following list of sets?
    {Blue, Red, Orange}, {Orange, Blue, Red}, {Red, Blue, Orange}, {Orange, Blue, Red} (Page 260) (Solution 11)

12. How many permutations and combinations are there in the following list of sets?
    {D, R, P}, {P, R, D}, {R, P, D}, {D, P, R} (Page 260) (Solution 12)

13. Given a total of 5 letters, how many permutations of these 5 letters are there, if letters are allowed to repeat? (Page 261) (Solution 13)

14. Given a total of 5 letters, how many permutations of 5 letters are there, if letters are allowed to repeat? (Page 260) (Solution 14)

15. Given 7 different ice cream flavors. One scoop of ice cream will be added to a cone of ice cream. Then a second scoop of ice cream will be added to this cone. How many permutations are possible for these two scoops, if the same flavor is allowed to repeat?
    (Page 260) (Solution 15)

16. For the digits from 0-9, how many variations exist if one chooses 4 digits? Each variation must have a different order, but the digits are allowed to repeat.
    (Page 260) (Solution 16)

17. A password is chosen with 8 characters. Each character can be any lowercase letter from the English alphabet. A letter cannot be used more than once. How many possible passwords are there? (Page 261) (Solution 17)

18. True or False - If three sets have the same elements, with different order, then all three sets count as one combination. (Page 259) (Solution 18)

19. True or False - Different permutations have different order. (Page 260) (Solution 19)

20. Solve (a) $\binom{51}{51}$ (b) $\binom{27}{27}$ (Page 262) (Solution 20)

21. Solve (a) $\binom{180}{0}$ (b) $\binom{31}{0}$ (Page 262) (Solution 21)

22. What is the main difference between permutations and combinations? (Page 261) (Solution 22)

23. Does the notation $\binom{r+n-1}{r}$ describe combinations with repetition, or combinations without repetition? (Page 261) (Solution 23)

24. Given the following row in Pascal's triangle, what is the next row in Pascal's triangle?
    1 8 28 56 70 56 28 8 1 (Page 265) (Solution 24)

25. Solve these expressions (a) $\sum_{i=3}^{i=6} i$ (b) $\sum_{1=0}^{i=2} i^2$ (Page 266) (Solution 25)

26. Solve these expressions (a) $\sum_{i=2}^{i=4} i+5$ (b) $\sum_{i=0}^{i=2} 2i+3$ (Page 267) (Solution 26)

27. Find the expansion of $(x+y)^4$ (Page 267) (Solution 27)

28. Find the expansion of $(x+y)^5$ (Page 268) (Solution 28)

29. Find the expansion of $(2x+3y)^3$ (Page 268) (Solution 29)

30. Find the sixth term of the expansion of $(x+2y)^8$. Use zero-based counting to count the terms. (Page 268) (Solution 30)

9.6 Binomial Theorem Solutions

1. 30 meals
2. 24 variations
3. 12 variations
4. (a) 24 (b) 5,040
5. (a) 1 (b) 1
6. (a) 360 (b) 4

7. (a) 20 (b) 42
8. 2 Combinations
9. 3 Combinations
10. 3 Permutations
11. 4 Permutations
12. 4 Permutations, 1 combination
13. 120
14. 3,125
15. 49 Permutations
16. This means solving for permutations, with repetition. 10,000 Permutations
17. This means solving for permutations, without repetition. 7,893,600 Permutations
18. True
19. True
20. (a) 1 (b) 1
21. (a) 1 (b) 1
22. Different permutations have different order, but different combinations have different elements.
23. Combinations with repetition
24. 1 9 36 84 126 126 84 36 9 1
25. (a) 18 (b) 5
26. (a) 24 (b) 15
27. $x^4 + 4x^3y + 6x^2y^2 + 4x^1y^3 + y^4$
28. $x^5 + 5x^4y + 10x^3y^3 + 10x^2y^4 + 5xy^5 + y^5$
29. $8x^3 + 36x^2y + 54xy^2 + 27y^3$
30. $1792x^2y^6$

Chapter 10: Sequences and Series

OVERVIEW

The section of this chapter are:

10.1 Linear vs Nonlinear Sequences
10.2 Arithmetic Sequences
10.3 Recursive Sequences
10.4 Geometric Sequences
10.5 Series

In everyday life, things might seem random, like roofs and staircases, but they have a hidden order. Even in nature, like languages or light, there's underlying organization and patterns. For example, a puppy might seem chaotic, but its DNA follows specific sequences and patterns. Similarly, sounds from voices or instruments may seem random, but they have structured components. Sequences and series help us understand and analyze these patterns.

OBJECTIVES

By the end of the chapter, you will be able to:

- Describe the difference between linear and nonlinear sequences
- Define and work with an arithmetic sequence
- Discuss the general concept of a recursive sequence
- Define and work with a geometric sequence
- Understand a finite sequence vs. an infinite sequence
- Understand and describe a series as a sum of a sequence

10.1 Linear vs. Nonlinear Sequences

Straight lines and sequences share a tight link through concepts like slope and order. Slope and linearity highlights this connection. In a straight line, values progress predictably. The y-value increases consistently for every change in y, mirroring a linear sequence.

1, 2, 3, 4, 5, 6   1, 3, 5, 7, 9, 11, 13, 15

Just above, to the left, one can see a list of numbers. The difference between any two numbers in this list is 1. To save time one might want to count by odde numbers. Just above, to the right, one can see a list of numbers. The difference between any two numbers on this list is 2. These sequences can be graphed so that x or the independent variable is the element number. This is shown below.

Fig. 10.1: Linear Sequence

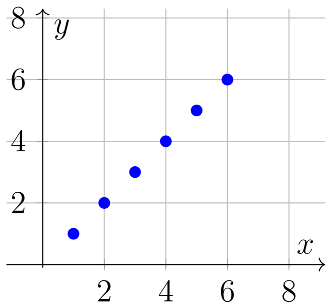
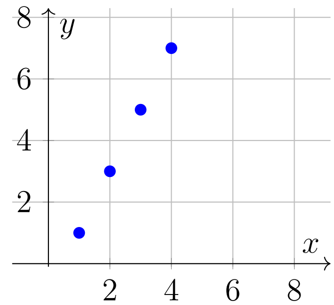


Fig. 10.2: Linear Sequence, Odds



One can see, just above, that linear sequences lead to straight lines.

1, 4, 7, 10, 13, 16   0, 5, 10, 15, 20, 25, 30, 35

The numbers listed above show a pattern: the difference between consecutive numbers is 3 on the left and 5 on the right. This consistent difference forms a pattern, akin to the straight lines graphed in Chapter 1. Thus, students have already encountered linear sequences. This chapter explores linear sequences through various descriptions and representations

In linear functions, the slope remains constant, while nonlinear functions exhibit varying slopes along curves. Similarly, in nonlinear sequences, the difference between numbers isn't consistent, leading to graphs that aren't simple straight lines.

Tip For a nonlinear sequence the difference between numbers in the sequence, is not constant.

10.2 Arithmetic Sequences

Instead of using broad terms like "order" or "pattern," one can delve into specifics when describing sequences. Each list above exhibits a distinct difference between its numbers or elements, known as the common difference. Sequences can be arranged in ascending or descending order, though it's typical to see them in ascending order.

Definition 10.1 — Arithmetic Sequence. A sequence of numbers, where the difference between any two numbers in the sequence has the same common difference. An arithmetic sequence is also a linear sequence.

$$\boxed{a_n = a_1 + d(n-1)}$$

Example:

0, 10, 20, 30, 40

The number in the sequence is a_n. The common difference is d, and n is the position of the number a_n in the sequence.

Tip An arithmetic sequence is simply a linear sequence of numbers.

Solved Problem 10.1 Given, the following arithmetic sequence, identify the common difference. 1, 9, 17, 25, 33, 41, 49

Since this is an arithmetic sequence, the difference between any two numbers is the common difference. For this sequence the common difference is $33-25=\boxed{8}$ ■

Solved Problem 10.2 Given, the following arithmetic sequence, identify the missing numbers.
1 7 $\boxed{}$ 19 25 $\boxed{}$ 37

Since, the common difference is 6, this can be used to find any missing numbers. The missing numbers are $\boxed{\text{13 and 31}}$. Notice that $7+6=13$ and $25+6=31$. ■

Solved Problem 10.3 Given, the starting number of 5, and a common difference of 4, what is the 6th number in an arithmetic sequence.

The formula $a_n = a_1 + d(n-1)$ is used here. The starting number a_1 is equal to 5. The common difference d is 4. The sixth number is needed so n is 6. The 6^{th} number a_n is then,

$$a_n = a_1 + d(n-1) = 5 + 4(6-1) = 5 + 4(5) = 4 + 20 = \boxed{25}$$ ■

One can see that an arithmetic sequence is simply a list of numbers with a common difference. The last solved problem asked about the 6^{th} number in a sequence. The solved problem before that involved a list of 7 numbers. With this in mind, an arithmetic sequence can be finite or infinite. An infinite arithmetic sequence is demonstrated below.

$$\{0, 3, 6, 9, 12, 15, ...\}$$

One can see, just above, that this sequence still has a common difference, but it continues indefinitely.

10.3 Recursive Sequences

A key criteria for an arithmetic sequence is that any two numbers in the sequence have the same common difference. This establishes a pattern. A recursive sequence is also defined by a key criteria.

Definition 10.2 — Recursive Sequence. In a recursive sequence, each element is defined by one or more terms before it. An element or term depends on a term before it, and this term before it also depends on a term before. This dependence and pattern repeats, recurs, or reoccurs, all the way back to the start of the sequence.

A Fibonacci sequence is an example of a recursive sequence. Each element is simply the sum of the previous two elements.

$\boxed{a_n = a_{n-1} + a_{n-2}}$

Fibonacci sequence example

0, 1, 1, 2, 3, 5, 8, 13, ...

In the definition above, a Fibonacci sequence is presented. In this sequence, each number is the sum of the two preceding ones. For instance, 5 is the sum of 3 and 2, and 8 is the sum of 3 and 5. While this pattern is straightforward, there's no fixed common difference between consecutive numbers, making it nonlinear. Plotting Fibonacci numbers on a graph reveals that the difference between them grows as the sequence progresses.

The difference between 1 and 2 is 1.

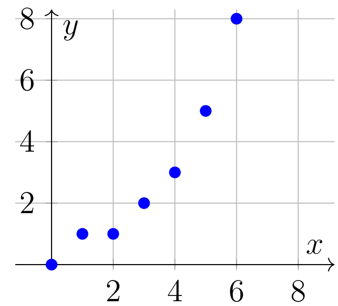
The difference between 3 and 5 is 2.

The difference between 5 and 8 is 3.

The difference between 8 and 13, is 5.

There is no longer a common difference between numbers.

Fig. 10.3: Fibonacci Sequence



The sixth Fibonacci number, with zero-based counting, is 8. To find the seventh Fibonacci number, you start with 0 and 1. Adding these two gives 1. Then, adding the last two numbers (1 and 1) gives 2, and so on. This repeating process is why they're called recursive sequences.

0 1   0 1 1   0 1 1 2   0 1 1 2 3   0 1 1 2 3 5   0 1 1 2 3 5 8

Solved Problem 10.4 Given, the following Fibonacci sequence, identify the missing numbers.

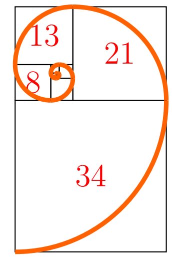
0 1 1 2 3 5 $\boxed{}$ 13 21 34 $\boxed{}$ $\boxed{}$

In the Fibonacci sequence any element depends on the two elements before it. It is then necessary to proceed from left to right. The first blank is $3+5=8$. The second blank is $21+34=55$. The third element is then $34+55=89$.

0 1 1 2 3 5 $\boxed{8}$ 13 21 34 $\boxed{55}$ $\boxed{89}$ ▪

The Fibonacci sequence can display some interesting and mysterious behavior. While keeping the focus on algebra, it is worthwhile to show and briefly discuss this behavior. An example of interesting properties of the Fibonacci sequence can be demonstrated with the Fibonacci Spiral that is demonstrated below.

Fig. 10.4: Fibonacci Spiral



Notice that 8, 13, 21, and 34 are Fibonacci numbers. This diagram consists of squares. The position of each next square rotates. Each square has a side with a dimension that is equal to a Fibonacci number.

In the Fibonacci sequence, another pattern emerges when dividing each Fibonacci number by its predecessor. For instance, dividing 8 by 5 yields 1.6. As one progresses to higher Fibonacci numbers, this division consistently approaches 1.618, often termed the Golden Ratio.

The golden ratio, often debated, appears in discussions on beauty, art, biology, and architecture. The Fibonacci Spiral, aesthetically pleasing, stems from Fibonacci numbers, which grow exponentially. The ratio between consecutive Fibonacci numbers approximates 1.618, termed the golden ratio or divine proportion, symbolized by the Greek letter phi (Φ). The Fibonacci sequence, starts with 0, 1, 1, 2, 3, 5, 8, ... The Fibonacci sequence extends to numbers like 144, 233, 377, and 600, with their corresponding ratios outlined below.

Table 10.1: Golden Ratio Φ

a_n	a_{n-1}	a_n/a_{n-1}
5	8	1.6
144	233	1.618
233	377	1.618
377	600	1.618
1,597	2,584	1.618

The table above shows that when you divide a Fibonacci number by its predecessor, the result is close to 1.618. Lower Fibonacci numbers like 5 and 8 yield a ratio of 1.6, while higher ones approach 1.618. Though often approximated as 1.618, the golden ratio can be expressed more precisely.

$\frac{233}{144}$ is approximated as 1.618 $\qquad$ $\frac{377}{233}$ is approximated as 1.618

A brief discussion regarding ratios, decimals, and approximations can be found on page 113.

10.4 Geometric Sequence

For an arithmetic sequence any two elements have a common difference or a constant difference. This is another way of saying that for any element a constant is **added**, to find the next term. This was demonstrated in Solved Problem 10.2.

A Geometric sequence, builds on this understanding. A difference is that for any element, the element is **multiplied** by a constant to find the next element.

Definition 10.3 — Geometric Sequence. A sequence of numbers, where for any element, the element is multiplied by a factor, or common ratio, to find the next element. A geometric sequence is a nonlinear sequence.

$$\boxed{a, ar, ar^2, \ldots \quad a_n = ar^{n-1}}$$

Example:
1, 3, 9, 27, 81, ...

Then a is called the starting value or the scaling factor. Then r is called the common ratio. This common ratio may or may not be a whole number.

The addition in the arithmetic sequence makes it a linear sequence. For a geometric sequence there is a common factor between any two elements. This multiplication makes it a nonlinear sequence, or simply a curved sequence.

Solved Problem 10.5 Given, the following geometric sequence, identify the scaling factor and the common ratio. 1 4 16 64 256

The scaling factor is also the first number, which is $\boxed{1}$. Then $16/4 = 4$, so the common ratio is $\boxed{4}$. This means that each element is multiplied by 4 to find the next element. ■

Solved Problem 10.6 Given, the following geometric sequence, identify the missing elements.
2 -12 $\boxed{}$ -432 $\boxed{}$ -15,552

Dividing any element by the element before it gives the common ratio. Then $12/2 = -6$. The common ratio of 6 is then used to find the missing elements. For example $-12 \times -6 = 72$, and $-432 \times -6 = 2,592$. The sequence is then 2 -12 $\boxed{72}$ -432 $\boxed{2{,}592}$ -15,552 ■

It was emphasized in the definition that the factor or common ratio may not be a whole number.

Solved Problem 10.7 Given the starting number of 8, what is the fifth number if the common ratio is $\frac{1}{2}$.

The formula $a_n = ar^{n-1}$ is used. The starting number or scaling factor a is 8. Then, for the fifth number $n = 5$.

$$a_5 = 8\left(\frac{1}{2}\right)^{5-1} = 8\left(\frac{1}{2}\right)^{4} = 8 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = 8 \times \frac{1}{16} = \frac{8}{16} = \boxed{\frac{1}{2}}$$

10.5 Series

This chapter has demonstrated an arithmetic sequence, a Fibonacci Sequence, and a geometric sequence. A key difference between a sequence and a series is that a series is the addition of terms. It has been mentioned that a sequence can be finite, meaning that it has a constant or fixed number of elements. The difference between a sequence and a series is demonstrated below.

Sequence A = 2, 4, 6, 8, 10, 12   Then, an element of A is a_n.

The sequence A above is an arithmetic sequence with a common difference of 2. In this sequence, a_2 represents the second element, which is 4. It's a finite sequence with only 6 elements. Below, a series based on this sequence is shown.

$$\sum_{n=1}^{n=6} a_n = 2 + 4 + 6 + 8 + 12 = 32$$

Both the sequence and the series contain the same finite set of numbers, following an arithmetic pattern. The primary distinction lies in the series involving addition. The series, displayed above, employs summation notation with upper and lower boundaries $n = 6$ and $n = 1$ respectively, as discussed in Definition 9.7. Now, let's explore a geometric sequence.

Sequence B = 1, 3, 9, 27, 81   Then, an element of B is b_n.

In the geometric sequence above, it's evident that the common factor or ratio between elements is 3, and the sequence is finite. Summing the numbers of this sequence results in the finite geometric series depicted below. As reiterated, a series involves adding the numbers in a sequence

$$\sum_{n=1}^{n=5} a_n = 1 + 3 + 9 + 27 + 81 = 121$$

Solved Problem 10.8 Evaluate the following series $0+1+1+2+3+5+8$

One can see right away that one is simply adding terms in a sequence. That is all that a series is. This finite series has a Fibonacci pattern, but one only needs to add the terms. $0+1+1+2+3+5+8 = \boxed{20}$

Solved Problem 10.9 Evaluate the following series $\sum_{i=1}^{i=5} 2i$

$\sum_{i=1}^{i=5} \boxed{2i}$ $\quad \sum_{i=1}^{i=5} 2i = 2\boxed{(1)} + 2\boxed{(2)} + 2\boxed{(3)} + 2\boxed{(4)} + 2\boxed{(5)}$

$$\sum_{i=1}^{i=5} 2i = 2+4+6+8+10 = \boxed{30}$$

Summation notation describes the addition of terms. The pattern of each term is shown, just above, with an orange border. One can see that each term will have i, and one can see that i is always multiplied by 2. Also, i will vary from 1 to 5. ■

In the last problem, we found an arithmetic pattern within a series and sequence. A series sums up a sequence, while a geometric sequence involves multiplying each element by a common ratio. In the solution, the compact summation notation expands, showing i with an orange border in each term. This emphasizes i's systematic increase in each term, crucial for understanding the pattern.

An arithmetic series, and a geometric series has been demonstrated. Then, there is the concept of an infinite series. This, again, builds on concepts that have been demonstrated and discussed. What if one is asked to find the the sum of the following arithmetic series.

$$\sum_{i=1}^{\infty} 2i = 2+4+6+8+10+12+... = \infty$$

Sure, one could add the first three terms, or one could add the first 5 terms. One could not find the sum of all the terms, simply because the terms never end. One can see that the sum would grow without end, so that it advances towards infinity, ∞. This is said to be a divergent series. Does it seem like each term is growing or approaching zero?

Definition 10.4 — Divergent Series. In a divergent series the sum does not approach a finite specific number or limit. Another characteristic of a divergent series is that **the terms do not approach zero**.

In the last infinite series, the upper boundary of the index of addition i was infinity, ∞. This does not always mean that the infinite series is divergent. To better understand a divergent series it is helpful to see the contrast between a divergent series and a convergent series.

$$\frac{1}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{8}+\frac{1}{13}... = \boxed{3.36}$$

There are multiple patterns in the series shown, just above. One pattern is that the denominators are Fibonacci numbers. The term $1/2$ is less than $1/1$. Then, the term $1/3$ is less than $1/2$. Each term is approaching zero, and this occurs when one is working with an infinite series that does lead to a finite, specific sum. This would be a finite sum. For this series, shown, just above, one can confirm that many terms can be added, and the sum will still not exceed 3.36.

Definition 10.5 — Convergent Series. A convergent series approaches a specific number or limit. A characteristic of a convergent series is that **the terms must approach zero**.

Another example of an infinite series is shown below.

$$\sum_{i=1}^{\infty} \frac{1}{4^i} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots = \boxed{\frac{1}{3}}$$

The series has an infinite number of terms, but it has a finite sum that is 1/3. Again, each denominator increases, which causes each term to decrease. The denominator 4 becomes the denominator 16. Then the denominator 64 becomes the denominator 256. One can can confirm that one can add many terms and the sum will not exceed 1/3.

10.6 Sequences and Series Problems

Click on the page number to the right to see the chapter discussion. Click on the solution number to navigate to the solution.

1. Which of the following sequences is nonlinear? (Page 273) (Solution 1)
   (a) 2, 5, 8, 11 (b) 3, 5, 7, 9 (c) 1, 4, 16, 64

2. Which of the following sequences is linear? (Page 273) (Solution 2)
   (a) Fibonacci sequence (b) Arithmetic sequence (c) Geometric sequence

3. True or False - An arithmetic sequence has a common ratio. (Page 274) (Solution 3)

4. True or False - A geometric sequence has a common difference.
   (Page 278) (Solution 4)

5. What is the common difference in the following sequence? (Page 275) (Solution 5)
   3, 7, 11, 15

6. Identify the missing numbers in the following sequence. (Page 275) (Solution 6)
   0, 12, $\boxed{}$, 36, 48, $\boxed{}$

7. Identify the missing numbers in the following sequence. (Page 278) (Solution 7)
   1, 5, $\boxed{}$, 125, 625, $\boxed{}$

8. What is the 5^{th} number in an arithmetic sequence if the first number is 3 and the common difference is 4? (Page 275) (Solution 8)

9. What is the 6^{th} number in an arithmetic sequence if the first number is 7 and the common difference is 2? (Page 275) (Solution 9)

10. Name an example of a recursive sequence. (Page 275) (Solution 10)

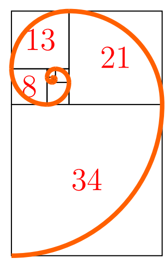
11. True or False - The Fibonacci sequence is a nonlinear sequence.
(Page 277) (Solution 11)

12. True or False - A nonlinear sequence does not have a common difference.
(Page 273) (Solution 12)

13. What would be the next two numbers in the following Fibonacci sequence?
610, 987, 1597, $\boxed{}$, $\boxed{}$ (Page 276) (Solution 13)

14. What is the ratio of the Fibonacci number 4,181 to the Fibonacci number before it 2,584?
(Page 277) (Solution 14)

15. What is the name of the spiral shown in this diagram? (Page 276) (Solution 15)



16. What is a common symbol for the golden ratio? (Page 277) (Solution 16)

17. Given the following geometric sequence, find the (a) scaling factor and the (b) common ratio. 2, 4, 8, 16, 32 (Page 278) (Solution 17)

18. Given the following geometric sequence, find the (a) scaling factor and the (b) common ratio. 3, 12, 48, 192 (Page 278) (Solution 18)

19. True or False - In a geometric series the scaling factor is equal to the starting value.
(Page 278) (Solution 19)

20. What is the 4^{th} number in a geometric sequence, if the first number is 2, and the common ratio is 3? (Page 278) (Solution 20)

21. What is the 7^{th} number in a geometric sequence, if the first number is 5, and the common ratio is 2? (Page 278) (Solution 21)

22. Evaluate $\sum_{i=1}^{i=4} i^3$
    (Page 279) (Solution 22)

23. Evaluate $\sum_{i=2}^{i=5} i+5$
    (Page 279) (Solution 23)

24. Evaluate $\sum_{i=0}^{i=5} i!$
    (Page 280) (Solution 24)

25. Evaluate $\sum_{k=0}^{k=3} 3k!$
    (Page 280) (Solution 25)

26. Evaluate $\sum_{n=0}^{n=5} (-1)^n$ (Page 280) (Solution 26)

27. Is the following series convergent or divergent? $\sum_{i=0}^{\infty} i^2$ (Page 280) (Solution 27)

28. Is the following series convergent or divergent? $\frac{1}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{8}+\frac{1}{13}+\frac{1}{21}+\ldots$
    (Page 281) (Solution 28)

29. Evaluate $\sum_{i=0}^{\infty} 2i^2+3$
    (Page 280) (Solution 29)

30. Evaluate $\sum_{i=0}^{\infty} \frac{1}{2^i}$
    (Page 280) (Solution 30)

10.7 Sequences and Series Solutions

1. Sequence c
2. Sequence b
3. False
4. False
5. 4
6. 24, 60
7. 25, 3,125
8. 17
9. 17
10. Fibonacci Sequence
11. True
12. True
13. 2,584 and 4,181
14. 1.618
15. Fibonacci Spiral
16. Phi, Φ
17. (a) 2 (b) 2
18. (a) 3 (b) 4
19. True
20. 54
21. 320
22. 100
23. 34
24. 154
25. 30
26. Zero
27. Divergent
28. Convergent
29. ∞
30. 2

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Appendix A: Fraction Operations

A.1 Simplifying Fractions

When adding, subtracting, multiplying, and dividing, it is often necessary to simplify fractions. Also, simplifying fractions includes practice with factoring. Factoring in different forms is useful throughout an algebra course. Consider the fraction below.

$$\frac{20}{5} = \frac{5 \times 4}{5} = \frac{(5)4}{(5)} \qquad \frac{\cancel{(5)}4}{\cancel{(5)}} = \boxed{4}$$

This fraction, just above, has a numerator of 20 and a denominator of 5. The denominator is rewritten as 5×4. Why is that?

Factors of 20 are integers that can be multiplied to produce 20. This is shown below.

Factors of 20 includes: 1, 2, 4, 5, 10, 20

Notice that any number in the list of factors, just above, can be multiplied by another number in the list above to produce 20. This is shown below. To find the factors one can start with 1, and then one lists integers that can be multiplied to produce 20.

$$1 \times 20 = 20 \quad 2 \times 10 = 20 \quad 4 \times 5 = 20$$

The fraction 20/5 does not include negative integers, but negative factors can be included.

Factors of 20 includes: $\pm 1, \pm 2, \pm 4, \pm 5, \pm 20$

The symbol $\pm$ is the "plus-or-minus" symbol, and it simply means plus or minus. So, ± 3 means 3 or -3. Likewise, ± 120 means 120 or -120. The simple phrase "120 or -120", may be confusing, at first. It is simply stating that one will need one of 2 numbers, either 120 or -120. After this is understood, one can confirm the factors, as shown below.

$$-1 \times -20 = 20 \quad -2 \times -10 = 20 \quad -4 \times -5 = 20$$

Finding factors, as shown above, makes it possible to simplify fractions. Both finding the factors, and simplifying the fractions should be applied and practiced.

Solved Problem A.1 List the positive factors of each integer. (a) 100 (b) 64

Recall that factors must be integers.

(a) Positive factors of 100 are 1, 2, 4, 5, 10, 20, 25, 50, 100

(b) Positive factors of 64 are 1, 2, 4, 8, 16, 32, 64 ■

Solved Problem A.2 List the factors of each expression. (a) x^3 (b) x^2y

---

Recall that factors must be integers.

(a) Factors of x^3 are $1, x, x^2, x^3$

(b) Factors of x^2y are $1, x, x^2, y, x^2y$ ■

Solved Problem A.3 Simplify the following fractions.

(a) $\dfrac{1000}{10}$ (b) $\dfrac{121}{11}$ (c) $\dfrac{x^3y}{x^2}$ (d) $\dfrac{72x^2y}{8x}$

(a) $\dfrac{1000}{10} = \dfrac{\cancel{10} \times 10 \times 10}{\cancel{10}} = \dfrac{10 \times 10}{1} = \boxed{100}$

(b) $\dfrac{121}{33} = \dfrac{\cancel{11} \times 11}{\cancel{11} \times 3} = \boxed{\dfrac{11}{3}}$

The factors of 121 are 1, $\boxed{11}$, 121. The factors of 33 are 1, 3, $\boxed{11}$, 33. The greatest common factor is then 11. The numerator is written in terms of the common factor, to arrive at 11×11. The denominator is also written in terms of 11, to arrive at 11×3. The 11 cancels out in the numerator and denominator, to leave the final answer.

(c) $\dfrac{x^3y}{x^2} = \dfrac{\cancel{x^2} \times xy}{\cancel{x^2}} = \boxed{\text{xy}}$

The factors of x^3y are $1, x, \boxed{x^2}, x^3, y$. The factors of x^2 are $1, x, \boxed{x^2}$. The numerator is written in terms of x^2, to arrive at $x^2 \times xy$. The denominator is written in terms x^2, to arrive at x^2. The greatest common factor x^2 cancels out to leave the final answer.

(d) $\dfrac{72x^2y}{8x} = \dfrac{\cancel{8x} \times 9xy}{\cancel{8x}} = \boxed{\text{9xy}}$

Both in the numerator and the denominator, one can find factors of the integers. Then one can find factors of the variables.

For the numerator, the factors of 72 are $1, 2, 4, \boxed{8}, 9, 12, 18, 36, x, x^2$. Then again, for the numerator the factors of x^2y are $1, \boxed{\text{x}}, x^2, y$. For the denominator, the factors of 8 are $1, 2, 4, \boxed{8}$. Then again, for the denominator the factors of x are $1, \boxed{\text{x}}$.

For the integers, the greatest common factor is 8. For the variables, the greatest common factor is x. The greatest common factor of both the numerator and the denominator is $8x$.

The numerator is written in terms of $8x$, to arrive at $8x \times 9xy$. The denominator is written in terms of $8x$, to arrive at $8x$. The greatest common factor cancels out, to leave the final answer. ▪

A.2 Adding and Subtracting Fractions

Definition A.1 — Fraction. A fraction is simply a piece, part, or portion of a whole. A fraction can be described as a ratio of the numerator to the denominator. A fraction can also be described as a quotient, in that the numerator is divided by a denominator.

$$\frac{3}{8} \qquad \begin{array}{l} 3 \text{ is the numerator} \\ 8 \text{ is the denominator} \end{array}$$

For a simple fraction, both the numerator and the denominator are integers.

An equivalent fraction can be created by taking a fraction and multiplying both the numerator and denominator by the same factor. This is demonstrated below.

$$\frac{3}{8} \qquad \frac{3\times 2}{8\times 2} = \boxed{\frac{6}{16}} \qquad \frac{3\times 4}{8\times 4} = \boxed{\frac{12}{32}} \qquad \frac{3\times 7}{8\times 7} = \boxed{\frac{21}{56}}$$

All three fractions, that are shown above, in an orange border, are equivalent. These equivalent fractions can also be simplified by reversing this procedure. These same fractions are simplified below.

$$\frac{6\div 2}{16\div 2} = \boxed{\frac{3}{8}} \qquad \frac{12\div 4}{32\div 4} = \boxed{\frac{3}{8}} \qquad \frac{21\div 7}{56\div 7} = \boxed{\frac{3}{8}}$$

The fractions (21/56) and (3/8) are equivalent fractions, but the fraction (3/8) is simplified. In order to simplify a fraction one must be able to divide the numerator and the denominator by the same number. As shown, just above, the fraction (6/16) is simplified by dividing both the numerator and the denominator by 2. Likewise, the fraction (12/34) is simplified by dividing both the numerator and the denominator by 4. While simplifying, when one is dividing the numerator, this should yield a whole number. Likewise, while simplifying, when one is dividing the denominator, one should arrive at a whole number.

Solved Problem A.4 Given the fraction $\frac{2}{5}$, find 3 other equivalent fractions.

This fraction can be multiplied by any 3 factors, to create equivalent fractions. One chooses any three factors, such as 2, 3, and 4.

$$\frac{2\times 2}{5\times 2} = \boxed{\frac{4}{10}} \qquad \frac{2\times 3}{5\times 3} = \boxed{\frac{6}{15}} \qquad \frac{2\times 4}{5\times 4} = \boxed{\frac{8}{20}}$$

All 3 fractions, shown in an orange border are equivalent. ■

Solved Problem A.5 Given the fractions $\frac{3}{4}$ and $\frac{3}{20}$, change one of the fractions, so that both fractions have the same denominator.

The denominator 4 can be made equal to 20, if it is multiplied by 5.

$\frac{3\times 5}{4\times 5} = \boxed{\frac{15}{20}}$ The fractions $\frac{15}{20}$ and $\frac{3}{4}$ are equivalent fractions. The 2 fractions $\frac{15}{20}$ and $\frac{3}{20}$, now have the same denominator, also called a common denominator. Notice that the lesser denominator is multiplied by a factor to equal the greater denominator. ■

Solved Problem A.6 Simplify the following fractions. (a) $\frac{20}{100}$ (b) $\frac{40}{64}$ (c) $\frac{4}{13}$

(a) $\frac{20\div 20}{100\div 20} = \boxed{\frac{1}{5}}$ (b) $\frac{40\div 8}{64\div 8} = \boxed{\frac{5}{8}}$ (c) Already simplified ■

Being able to modify a fraction so that it has a particular denominator, was demonstrated in the last solved problem. This makes it possible to add and subtract fractions. In order to add or subtract fractions, the two fractions need to have a common denominator. Two approaches, can be applied to find a common denominator.

- Modify the lesser denominator to match the greater denominator
- Multiply both denominators to make a new denominator

Solved Problem A.7 Solve $\frac{3}{12} + \frac{1}{3}$

These two fractions have different denominators. The greater denominator 12 is a multiple of 3. The lesser denominator 3 can be modified to equal 12. Then the two fractions can be added. $\frac{1\times 4}{3\times 4} = \frac{4}{12}$ $\quad \frac{3}{12} + \frac{4}{12} = \frac{3+4}{12} = \boxed{\frac{7}{12}}$. ■

Solved Problem A.8 Solve $\dfrac{1}{7}+\dfrac{3}{5}$

The greater denominator 7 is not a multiple of 5. These denominators can be multiplied to arrive at a common denominator of 35. For both fractions, one finds equivalent fractions with a denominator of 35.

$$\frac{1\times 5}{7\times 5}=\boxed{\frac{5}{35}} \qquad \frac{3\times 7}{5\times 7}=\boxed{\frac{21}{35}} \qquad \frac{5}{35}+\frac{21}{35}=\boxed{\frac{26}{35}}$$

In order to subtract fractions, again, both fractions need to have the same denominator, or a common denominator. A common denominator is found in the same way, as for addition of fractions.

Solved Problem A.9 Solve $\dfrac{10}{26}-\dfrac{1}{13}$

The denominator 26 is a multiple of the denominator 13. So, 26 is a common denominator of both fractions. $\dfrac{1\times 2}{13\times 2}=\dfrac{2}{26} \qquad \dfrac{10}{26}-\dfrac{2}{26}=\boxed{\dfrac{8}{26}}$

Solved Problem A.10 Solve $\dfrac{8}{9}-\dfrac{1}{4}$

The denominator 9 is not a multiple of 4. These denominators can be multiplied to find the common denominator of 36. $\dfrac{8\times 4}{9\times 4}=\boxed{\dfrac{32}{36}} \qquad \dfrac{1\times 9}{4\times 9}=\boxed{\dfrac{9}{36}} \qquad \dfrac{32}{36}-\dfrac{9}{36}=\boxed{\dfrac{23}{36}}$

A.3 Multiplying and Dividing Fractions

Multiplying fractions does not require a common denominator. This means that one can simply multiply the numerators, and then one multiplies the denominators. This is demonstrated below.

$$\frac{1}{10}\times\frac{5}{3}=\frac{1\times 5}{10\times 3}=\boxed{\frac{5}{30}} \qquad\qquad \frac{5}{8}\times\frac{7}{5}=\frac{5\times 7}{8\times 5}=\boxed{\frac{35}{40}}$$

When fractions are multiplied it may be necessary to simplify the result. Simplification of fractions was demonstrated in Solved Problem A.6.

Solved Problem A.11 Solve (a) $\dfrac{12}{5}\times\dfrac{5}{12}$ (b) $\dfrac{7}{9}\times\dfrac{3}{4}$

(a) $\frac{12}{5} \times \frac{5}{12} = \frac{12 \times 5}{5 \times 12} = \frac{60}{60} = \boxed{1}$ (b) $\frac{7}{9} \times \frac{3}{4} = \frac{7 \times 3}{9 \times 4} = \frac{21}{36} = \frac{21 \div 3}{36 \div 3} = \boxed{\frac{7}{12}}$ ■

One can see in this last solved problem that each solution was simplified. For the first exercise in the last solved problem, multiplying numerators and denominators, leads to 60/60. This fraction is the correct product and result of this multiplication exercise. However, it is common practice, and it is best to simplify a fraction if possible. Recall that for any fraction or ratio, if the numerator is equal to the denominator, then it is equal to 1.

In the last solved problem, in the second exercise, again 21/36 is equal to the result of the multiplication. Again, it is best to simplify the fraction. It is possible to divide both the numerator and the denominator by the same number, 3. This leads to the simplified result.

Dividing fractions is quite similar to multiplying fractions. A difference is that, while dividing, one makes use of the reciprocal of the second fraction. This is demonstrated below.

$$\frac{12}{16} \div \boxed{\frac{3}{1}} = \frac{12}{16} \times \boxed{\frac{1}{3}} \qquad \frac{12 \times 1}{16 \times 3} = \frac{12}{48} = \frac{12 \div 12}{48 \div 12} = \boxed{\frac{1}{4}}$$

One can see just above that division of fractions becomes multiplication. The division by (3/1) becomes multiplication by (1/3). The fraction (1/3) is the reciprocal of (3/1). This multiplication leads to (12/48), which is the simplified to (1/4). The following steps are helpful while learning to divide one fraction by another...

Step 1: Find the reciprocal of the second fraction
Step 2: Multiply the first fraction by this reciprocal.

Solved Problem A.12 Find the reciprocal of each fraction (a) $\frac{7}{3}$ (b) $\frac{13}{8}$ (c) $\frac{9}{5}$

(a) $\frac{3}{7}$ (b) $\frac{8}{13}$ (c) $\frac{5}{9}$ ■

Solved Problem A.13 Solve (a) $\frac{40}{100} \div \frac{4}{2}$ (b) $\frac{6}{9} \div \frac{3}{1}$

(a) $\frac{40}{100} \div \frac{4}{2} = \frac{40}{100} \times \frac{2}{4} = \frac{40 \times 2}{100 \times 4} = \frac{80}{400} = \frac{80 \div 80}{400 \div 80} = \boxed{\frac{1}{5}}$

(b) $\frac{6}{9} \div \frac{3}{1} = \frac{6}{9} \times \frac{1}{3} = \frac{6 \times 1}{9 \times 3} = \frac{6}{27} = \frac{6 \div 3}{27 \div 3} = \boxed{\frac{2}{9}}$ ■